On a one-dimensional quadratic operator pencil with a small periodic perturbation

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Abstract

We consider a quadratic operator pencil with a small periodic perturbation multiplied by the spectral parameter. It is motivated, in particular, by a one-dimensional Klein-Gordon equation with a time-parity-symmetric perturbation. We study in details the structure of the considered operator pencil. We show that its essential spectrum has a band structure and at certain thresholds, the bands bifurcate into small parabolas. We then study how the isolated limiting eigenvalues behave under the perturbation. We show that if zero is a limiting isolated eigenvalue, under the perturbation it remains an eigenvalue but an additional isolated eigenvalue can emerge from zero. The most part of the paper is devoted to studying the isolated eigenvalues converging to the essential spectrum. We establish sufficient conditions for the existence and absence of such eigenvalues and in the case of the existence, we calculate the leading terms of their asymptotic expansions.

Keywords: quadratic operator pencil, periodic perturbation, band spectrum, isolated eigenvalue, asymptotics

1 Introduction

In the present work we consider a one-dimensional quadratic operator pencil with a small periodic perturbation:

$$\mathcal{H}_\varepsilon(\lambda) = -\frac{d^2}{dx^2} + V + i\varepsilon\lambda\gamma + \varkappa_* - \lambda^2 \quad \text{on} \quad \mathbb{R}.$$

Here ε is a small positive parameter, V is a real potential decaying exponentially at infinity, γ is an odd real 1-periodic function, $\varkappa_*$ is a real constant, λ is a spectral parameter. The main motivation as well as the main application for such model is the perturbed one-dimensional Klein-Gordon equation

$$u_{tt} - u_{xx} + \varepsilon\gamma(x)u_t + F(u) = 0, \quad t > 0, \quad x \in \mathbb{R}. \tag{1.1}$$

The perturbative viscous-like term $\varepsilon\gamma u_t$ describes loss and gain in the system and these gain and loss are balanced in the sense that function γ is odd. The system itself is

parity-time-symmetric, namely, the change of variables $x \mapsto -x$, $t \mapsto -t$ keeps equation (1.1) unchanged.

Function F describes a non-linearity in the equation and we assume that the equation

$$-\phi_{xx} + F(\phi) = 0, \quad x \in \mathbb{R},$$

has a static kink solution $\phi = \phi(x)$. We also assume that

$$\lim_{x\to\pm\infty} F'(\phi(x)) = \varkappa_*,$$

where $\varkappa_* \in \mathbb{R}$ is some constant.

Let us consider the stability of kink ϕ. Namely, we consider a small dynamical perturbation of ϕ:

$$u(x,t) = \phi(x) + e^{i\lambda t}\psi(x), \quad \lambda = const, \quad \lambda \in \mathbb{C},$$

and linearize equation (1.1) w.r.t. ψ. The linearized equation is the eigenvalue problem for our operator pencil

$$-\psi_{xx} + (i\varepsilon\lambda\gamma + F'(\phi) - \lambda^2)\psi = 0, \quad x \in \mathbb{R},$$

where potential V is defined as $V = F'(\phi) - \varkappa_*$.

Operator pencils appearing in the above described stability analysis for nonlinear equation like (1.1) were recently studied in [6], [10]. In [10], operator pencil $\mathcal{H}_\varepsilon$ was considered in the case, when γ is smooth and decays exponentially at infinity. A similar two-dimensional operator pencil was addressed in [6]. The main results of [6], [10] state that small perturbative term $i\varepsilon\lambda\gamma$ in the considered operator pencil with a localized γ generates eigenvalues and/or resonances bifurcating from the spectral points of the limiting spectrum (corresponding to the case $\varepsilon = 0$).

This phenomenon, eigenvalues and resonances bifurcating from the edges of the essential spectrum, has a long history and it was studied both by mathematicians and physicians. For Schrödinger operators with a localized potential multiplied by a small coupling constant such phenomenon was studied, for instance, in [19], [20], [21], [29]. More general perturbations of one- and two-dimensional operators were treated in [16], [17]. Periodic operators with small localized perturbations described both by potentials and more general operators were studied in [2], [3], [18]. More complicated models like quantum and acoustic waveguides with localized perturbations were considered by many authors, see, for instance, [5], [7], [8], [9], [11], [12], [13], [14], [15], [22], [24], [25], [26], [27], [28], see also the references therein and other works by these authors. The main obtained results provide the necessary and sufficient conditions for the existence and absence of the emerging eigenvalues and in the case of existence, various asymptotic estimates and asymptotic expansions were obtained. However, to the best of our knowledge, the influence of a small periodic perturbation on the emergence of the eigenvalues and resonances was not considered before.

The present paper is a continuation of both the cited papers on operators pencils related to nonlinear equation (1.1) and of the cited papers on the emerging eigenvalues and the resonances. Namely, we study in details the structure of the spectrum of operator pencil $\mathcal{H}_\varepsilon$. We show that the residual spectrum is empty and there is only the essential and point components. For the essential spectrum we prove that it has a band structure and describe the shape of the bands for sufficiently small ε. We find that in certain points the bifurcation of the bands occurs and the bands have the form of small parabolas. The most part of the paper is devoted to the studying of the isolated eigenvalues of $\mathcal{H}_\varepsilon$. Perturbation of the limiting isolated eigenvalues is in fact done on the base of the regular perturbation theory. At the same time, if zero is a limiting isolated eigenvalue, it behaves quite irregularly under the presence of the term $i\varepsilon\lambda\gamma$: this zero eigenvalue is still an eigenvalue of the perturbed operator pencil and in some cases there exists an additional eigenvalue converging to zero. Apart from perturbing

limiting isolated eigenvalues, we also consider the isolated eigenvalues converging to the essential spectrum. We find that they can converge only to certain thresholds in the essential spectrum and near each threshold there can none or one or two isolated eigenvalues. Sufficient conditions for the existence and absence of such eigenvalues are provided and in the case of the existence, the leading terms of the asymptotic expansions are obtained.

2 Problem and main results

We introduce the self-adjoint Schrödinger operator

$$\mathcal{H}_0 := -\frac{d^2}{dx^2} + V,$$

in $L_2(\mathbb{R})$ on domain $W_2^2(\mathbb{R})$. Here $V \in C(\mathbb{R})$ is a real function decaying at infinity:

$$|V(x)| \leqslant Ce^{-\vartheta|x|}, \tag{2.1}$$

where C, ϑ are some positive constants independent of x. The main object of our study is the quadratic operator pencil

$$\mathcal{H}_\varepsilon(\lambda) := \mathcal{H}_0 + i\varepsilon\lambda\gamma + \varkappa_* - \lambda^2 \quad \text{in} \quad L_2(\mathbb{R}).$$

Here $\gamma \in C(\mathbb{R})$ is a real odd 1-periodic function, $\varkappa_* \in \mathbb{R}$ is a fixed constant. In this paper we are interested in the behavior of the spectrum of operator pencil $\mathcal{H}_\varepsilon(\lambda)$ as $\varepsilon \to +0$.

We define the resolvent set of $\mathcal{H}_\varepsilon(\lambda)$ as the set of λ such that there exists a bounded inverse operator $\mathcal{H}_\varepsilon^{-1}(\lambda)$ in $L_2(\mathbb{R})$. The spectrum $\sigma(\mathcal{H}_\varepsilon)$ of $\mathcal{H}_\varepsilon(\lambda)$ is the complement to the resolvent set. The point spectrum $\sigma_{\mathrm{pnt}}(\mathcal{H}_\varepsilon)$ is introduced as the set of eigenvalues, and an eigenvalue of $\mathcal{H}_\varepsilon(\lambda)$ is a number $\lambda \in \mathbb{C}$ such that the equation $\mathcal{H}_\varepsilon(\lambda)\psi = 0$ has a non-trivial solution called an eigenfunction. The essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{H}_\varepsilon)$ is defined in terms of the characteristic sequences. Namely, $\lambda \in \sigma_{\mathrm{ess}}(\mathcal{H}_\varepsilon)$ if there exists a sequence $\psi_n \in W_2^2(\mathbb{R}^2)$, which is bounded and non-compact in $L_2(\mathbb{R})$ and $\mathcal{H}_\varepsilon(\lambda)\psi_n \to 0$ in $L_2(\mathbb{R})$ as $n \to \infty$. The residual spectrum $\sigma_{\mathrm{res}}(\mathcal{H}_\varepsilon)$ is defined as

$$\sigma_{\mathrm{res}}(\mathcal{H}_\varepsilon) := \sigma(\mathcal{H}_\varepsilon) \setminus \big(\sigma_{\mathrm{ess}}(\mathcal{H}_\varepsilon) \cup \sigma_{\mathrm{pnt}}(\mathcal{H}_\varepsilon)\big).$$

Let $\mathcal{T}$ be the operator of complex conjugation in $L_2(\mathbb{R})$: $\mathcal{T}u = \overline{u}$. Then it is obvious that operator pencil $\mathcal{H}_\varepsilon(\lambda)$ is $\mathcal{T}$-self-adjoint in the following sense:

$$(\mathcal{H}_\varepsilon(\lambda))^* = \mathcal{H}_\varepsilon(-\overline{\lambda}), \quad \mathcal{T}(\mathcal{H}_\varepsilon(\lambda))^* = \mathcal{H}_\varepsilon(\overline{\lambda})\mathcal{T}, \tag{2.2}$$

In particular, it implies immediately that $\big(\mathcal{H}_\varepsilon(-\overline{\lambda})\big)^{-1} = \mathcal{T}\big(\mathcal{H}_\varepsilon(\lambda)\big)^{-1}\mathcal{T}$ and

$$\big(\sigma(\mathcal{H}_\varepsilon)\big)^\dagger = \sigma(\mathcal{H}_\varepsilon), \quad \big(\sigma_{\mathrm{pnt}}(\mathcal{H}_\varepsilon)\big)^\dagger = \sigma_{\mathrm{pnt}}(\mathcal{H}_\varepsilon), \quad \big(\sigma_{\mathrm{ess}}(\mathcal{H}_\varepsilon)\big)^\dagger = \sigma_{\mathrm{ess}}(\mathcal{H}_\varepsilon), \tag{2.3}$$

where superscript † denotes the symmetric reflection w.r.t. the imaginary axis of a set in the complex plane: $M^\dagger := \{-\overline{\lambda} : \lambda \in M\}$. If, in addition, function V is even, then

$$\begin{aligned}
&\mathcal{P}(\mathcal{H}_\varepsilon(\lambda))^* = \mathcal{H}_\varepsilon(\overline{\lambda})\mathcal{P}, \quad \mathcal{P}\mathcal{T}\mathcal{H}_\varepsilon(\lambda) = \mathcal{H}_\varepsilon(\overline{\lambda})\mathcal{P}\mathcal{T},\\
&\big(\mathcal{H}_\varepsilon(\overline{\lambda})\big)^{-1} = \mathcal{P}\mathcal{T}\big(\mathcal{H}_\varepsilon(\lambda)\big)^{-1}\mathcal{P}\mathcal{T},
\end{aligned}$$

where $(\mathcal{P}u) := u(-x)$. Hence,

$$\big(\sigma(\mathcal{H}_\varepsilon)\big)^\ddagger = \sigma(\mathcal{H}_\varepsilon), \quad \big(\sigma_{\mathrm{pnt}}(\mathcal{H}_\varepsilon)\big)^\ddagger = \sigma_{\mathrm{pnt}}(\mathcal{H}_\varepsilon), \quad \big(\sigma_{\mathrm{ess}}(\mathcal{H}_\varepsilon)\big)^\ddagger = \sigma_{\mathrm{ess}}(\mathcal{H}_\varepsilon), \tag{2.4}$$

where superscript ‡ denotes the symmetric reflection w.r.t. the real axis of a set in the complex plane: $M^\ddagger := \{\overline{\lambda} : \lambda \in M\}$.

To formulate our main results, we first describe the spectrum of operator $\mathcal{H}_0$. Thanks to the assumptions for V, it is a self-adjoint lower semi-bounded operator in $L_2(\mathbb{R})$. Its residual spectrum is empty, the essential spectrum reads as

$$\sigma_{\mathrm{ess}}(\mathcal{H}_0) = [0, +\infty)$$

since potential $V(x)$ tends to zero as $x \to \pm\infty$. Since potential V decays exponentially fast at infinity, the discrete spectrum $\sigma_{\mathrm{dsc}}(\mathcal{H}_0)$ consists of finitely many (simple) discrete eigenvalues $\varkappa_j < 0$. The associated eigenfunctions normalized in $L_2(\mathbb{R})$ are denoted by Ψ_j, $\Psi_j = \Psi_j(x)$. The case of empty discrete spectrum is not excluded from the consideration.

In view of the above facts on operator $\mathcal{H}_0$, the residual spectrum of the operator pencil $\mathcal{H}_0 + \varkappa_* - \lambda^2$ is empty, the discrete spectrum consists of the isolated eigenvalues $\pm\sqrt{\varkappa_j + \varkappa_*}$. All these eigenvalues are simple and the associated eigenfunctions are Ψ_j.

The essential spectrum of $\mathcal{H}_0 + \varkappa_* - \lambda^2$ is a pair of curves $\pm\sqrt{\varkappa_* + t}$, $t \in [0, +\infty)$. Provided $\varkappa_* \geqslant 0$, these are just two half-lines $\sigma_{\mathrm{ess}}(\mathcal{H}_0 + \varkappa_* - \lambda^2) = (-\infty, -\sqrt{\varkappa_*}] \cup [\sqrt{\varkappa_*}, +\infty)$. As $\varkappa_* < 0$, the essential spectrum is a cross in the complex plane:

$$\sigma_{\mathrm{ess}}(\mathcal{H}_0 + \varkappa_* - \lambda^2) = \left\{\lambda = it : \ -\sqrt{|\varkappa_*|} \leqslant t \leqslant \sqrt{|\varkappa_*|}\right\} \cup \mathbb{R}.$$

Now we are in position to formulate our first main result.

Theorem 2.1. *The residual spectrum of $\mathcal{H}_\varepsilon(\lambda)$ is empty. The spectrum of $\mathcal{H}_\varepsilon(\lambda)$ satisfies*

$$\sigma(\mathcal{H}_\varepsilon)) \subseteq \left\{\lambda \in \mathbb{C} : \ \mathrm{dist}(\lambda^2 - \varkappa_*, \sigma(\mathcal{H}_0)) \leqslant \varepsilon|\lambda| \sup_{[-\pi,\pi]} |\gamma|\right\}. \tag{2.5}$$

The second result describes the eigenvalues of $\mathcal{H}_\varepsilon(\lambda)$ converging to the isolated eigenvalues of $\mathcal{H}_0 + \varkappa_* - \lambda^2$.

Theorem 2.2. *Assume that $\varkappa_j \neq \varkappa_*$ is a discrete eigenvalue of operator $\mathcal{H}_0$. Then there exists a pair of complex conjugate isolated eigenvalues $\lambda_\varepsilon^\pm$ of operator pencil $\mathcal{H}_\varepsilon(\lambda)$ converging to $\pm\sqrt{\varkappa_j + \varkappa_*}$ as $\varepsilon \to +0$. Both these eigenvalues are simple and these are the only points of spectrum $\sigma(\mathcal{H}_\varepsilon(\lambda))$ in the vicinity of the points $\pm\sqrt{\varkappa_j + \varkappa_*}$ for sufficiently small ε. If $\varkappa_j + \varkappa_* < 0$, then eigenvalues $\lambda_\varepsilon^\pm$ are pure imaginary. Eigenvalue $\lambda_\varepsilon^\pm$ and the associated eigenfunction $\psi_\varepsilon^\pm$ are holomorphic w.r.t. ε (that latter is holomorphic in the sense of $W_2^2(\mathbb{R})$-norm):*

$$\lambda_\varepsilon^\pm = \pm\sqrt{\varkappa_j + \varkappa_*} + \sum_{n=1}^{\infty} \varepsilon^n \Lambda_n^\pm, \quad \psi_\varepsilon^\pm = \Psi_j + \sum_{n=1}^{\infty} \varepsilon^n \psi_n^\pm, \tag{2.6}$$

$$\Lambda_1^\pm = \frac{i(\gamma\Psi_j, \Psi_j)_{L_2(\mathbb{R})}}{2}, \quad \Lambda_2^\pm = \frac{i(\gamma\psi_1, \Psi)}{2} \pm \frac{(\Lambda_1^\pm)^2}{2\sqrt{\varkappa_j + \varkappa_*}}, \tag{2.7}$$

where $\psi_1^\pm$ is the unique solution to the equation

$$(\mathcal{H}_0 - \varkappa_j)\psi_1^\pm = \pm\sqrt{\varkappa_j + \varkappa_*}(-i\gamma + 2\Lambda_1^\pm)\Psi,$$

obeying the condition $(\psi_1, \Psi_j)_{L_2(\mathbb{R})} = 0$. Other coefficients of series (2.6) are determined in Section 4. Eigenfunctions $\psi_\varepsilon^\pm$ are related by the identity

$$\psi_\varepsilon^- = \overline{\psi_\varepsilon^+}.$$

Suppose that $\varkappa_j = -\varkappa_*$ is a discrete eigenvalue of operator $\mathcal{H}_0$. Then $\lambda = 0$ is a simple isolated eigenvalue of operator pencil $\mathcal{H}_\varepsilon(\lambda)$ and the associated eigenfunction is Ψ_j. If

$$(\gamma\Psi_j, \Psi_j)_{L_2(\mathbb{R})} = 0, \tag{2.8}$$

then the eigenvalue $\lambda = 0$ is the only point of spectrum $\sigma(\mathcal{H}_\varepsilon(\lambda))$ in the vicinity of zero. If

$$(\gamma\Psi_j, \Psi_j)_{L_2(\mathbb{R})} \neq 0, \tag{2.9}$$

then there exists one more simple pure imaginary isolated eigenvalue λ_ε converging to zero as $\varepsilon \to +0$. This eigenvalue and the associated eigenfunction are holomorphic w.r.t. ε (that latter is holomorphic in the sense of $W_2^2(\mathbb{R})$-norm):

$$\begin{aligned}
&\lambda_\varepsilon = \varepsilon\Lambda_1 + \sum_{n=3}^{\infty} \varepsilon^n \Lambda_n, \quad \psi_\varepsilon = \Psi_j + \sum_{n=2}^{\infty} \varepsilon^n \psi_n, \\
&\Lambda_1 = i(\gamma\Psi_j, \Psi_j)_{L_2(\mathbb{R})}, \quad \Lambda_3 = i(\gamma\psi_2, \Psi)_{L_2(\mathbb{R})},
\end{aligned} \tag{2.10}$$

where ψ_2 is the unique solution to the equation

$$(\mathcal{H}_0 - \varkappa_j)\psi_2 = -i\Lambda_1\gamma\Psi_j,$$

obeying the condition $(\psi_2, \Psi_j)_{L_2(\mathbb{R})} = 0$. Other coefficients of series (2.10) are determined in Section 4.

In order to describe the essential spectrum of $\mathcal{H}_\varepsilon$, we employ Floquet-Bloch theory. Namely, we introduce an auxiliary operator pencil:

$$\mathcal{H}_\varepsilon^\tau(\lambda) := \mathcal{H}_0(\tau) + i\varepsilon\lambda\gamma - \lambda^2,$$

where

$$\mathcal{H}_0(\tau) := \left(i\frac{d}{dx} + \tau\right)^2, \quad \tau \in \left[-\frac{1}{2}, \frac{1}{2}\right) + \varkappa_*,$$

is a self-adjoint operator in $L_2(-\pi,\pi)$ subject to periodic boundary conditions.

We first describe the spectrum of operator pencil $\mathcal{H}_0^\tau(\lambda)$. The spectrum of this operator pencil is pure point. For each $\tau \in [-\frac{1}{2}, \frac{1}{2}]$, the eigenvalues of $\mathcal{H}_0^\tau$ are $\pm\sqrt{(n-\tau)^2 + \varkappa_*}$, $n \in \mathbb{Z}$, and the associated eigenfunctions are e^{inx}. Provided $\tau \neq 0$, $\tau \neq \pm\frac{1}{2}$, these eigenvalues are simple. As $\tau = 0$, the eigenvalues $\pm\sqrt{n^2 + \varkappa_*}$ are double and the associated eigenfunctions are $\cos nx$ and $\sin nx$. As $\tau = -\frac{1}{2}$, the eigenvalues $\pm\sqrt{\left(n - \frac{1}{2}\right)^2 + \varkappa_*}$ are also double and the associated eigenfunctions are e^{inx} and $e^{-i(n-1)x}$. As $\tau = \frac{1}{2}$, the eigenvalues $\pm\sqrt{\left(n + \frac{1}{2}\right)^2 + \varkappa_*}$ are also double and the associated eigenfunctions are e^{-inx} and $e^{i(n+1)x}$.

In Section 5 we shall prove the following auxiliary lemma.

Lemma 2.3. *For each ε and τ the essential and residual spectra of $\mathcal{H}_\varepsilon^\tau$ are empty, i.e., $\sigma(\mathcal{H}_\varepsilon^\tau) = \sigma_{\mathrm{pnt}}(\mathcal{H}_\varepsilon^\tau)$.*

Our next results describe the essential spectrum.

Theorem 2.4. *The essential spectrum of operator pencil $\mathcal{H}_\varepsilon(\lambda)$ is symmetric both w.r.t. the real and imaginary axes. It is given by the formula*

$$\sigma_{\mathrm{ess}}(\mathcal{H}_\varepsilon) = \bigcup_{n\in\mathbb{Z}} \left\{\lambda^{(n)}(\varepsilon,\tau) : \tau \in \left[-\frac{1}{2}, \frac{1}{2}\right)\right\}, \tag{2.11}$$

where $\lambda^{(n)}(\varepsilon,\tau)$ are the eigenvalues of operator pencil $\mathcal{H}_\varepsilon^\tau$.

Fix $n \in \mathbb{Z}$ and let $\lambda^{(n)}(0,\tau) := \pm\sqrt{(n-\tau)^2 + \varkappa_*}$. There exists a constant $C > 0$ such that for sufficiently small ε and τ satisfying

$$C\varepsilon \leqslant |\tau| \leqslant \frac{1}{2} - C\varepsilon, \tag{2.12}$$

there exist exactly one eigenvalue $\lambda^{(n)}(\varepsilon,\tau)$ of $\mathcal{H}_\varepsilon^\tau$ converging to $\lambda^{(n)}(0,\tau)$ as $\varepsilon \to +0$. This eigenvalue is simple and satisfies the asymptotics

$$\lambda^{(n)}(\varepsilon,t) = \lambda^{(n)}(0,\tau) - \varepsilon^2\lambda_0 \frac{(\gamma\psi_1, \psi_0)_{L_2(-\pi,\pi)}}{2} + O(\varepsilon^3), \tag{2.13}$$

where ψ_1 is the unique solution to the equation

$$\big(\mathcal{H}_0(\tau)+\varkappa_*-\lambda_0^2\big)\psi_1+\gamma\psi_0=0 \tag{2.14}$$

obeying the condition $(\psi_1,\psi_0)_{L_2(-\pi,\pi)}=0$. If $\lambda^{(n)}(0,\tau)$ is real, then $\lambda^{(n)}(\varepsilon,\tau)$ is real, too. If $\lambda^{(n)}(0,\tau)=0$, then $\lambda^{(n)}(\varepsilon,\tau)=0$.

In the next theorem we described the eigenvalues of $\mathcal{H}_\varepsilon^\tau$ as $|\tau|\leqslant C\varepsilon$ or $|\tau|\geqslant\frac{1}{2}-C\varepsilon$.

Theorem 2.5. *Fix $n\in\mathbb{Z}$ and let $\lambda^{(n)}(0,\tau):=\pm\sqrt{(n-\tau)^2+\varkappa_*}$. As $|\tau|\leqslant C\varepsilon$, where C comes from (2.12), operator pencil $\mathcal{H}_\varepsilon^\tau$ has exactly two eigenvalues $\lambda_\pm^{(n)}(\varepsilon,t)$ converging to $\lambda^{(n)}(0,0)$ as $\varepsilon\to+0$ uniformly in t. If $\lambda_0:=\lambda^{(n)}(0,0)$ is real, then the asymptotics for eigenvalues $\lambda_\pm^{(n)}(\varepsilon,t)$ are*

$$\begin{aligned}\lambda_\pm^{(n)}(\varepsilon,t)=&\lambda_0\pm\frac{i\varepsilon}{4\lambda_0}\sqrt{\lambda_0^2\alpha_0^2(n)-4t^2n^2+O(\varepsilon^2)}\\&+\frac{\varepsilon^2}{4\lambda_0}\left(\left(2-\frac{n^2}{2\lambda_0^2}\right)t^2-\frac{7\alpha_0^2(n)}{8}+\lambda_0^2\alpha_1(n)\right)+O(\varepsilon^3),\end{aligned} \tag{2.15}$$

$$\begin{aligned}&\alpha_0(n):=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\gamma(x)\sin 2nx\,dx,\\&\alpha_1(n):=\frac{1}{\sqrt{\pi}}\int\limits_{-\pi}^{\pi}\gamma(x)\big(u_+(x)\cos nx+u_-(x)\sin nx\big)\,dx,\end{aligned} \tag{2.16}$$

where $u_\pm$ are the solutions to the equations

$$\big(\mathcal{H}_0(0)-n^2\big)u_\pm=\big(\gamma\psi_0^\pm-\alpha_0(n)\psi_0^\mp\big) \tag{2.17}$$

orthogonal to $\cos nx$ and $\sin nx$ in $L_2(-\pi,\pi)$.

If λ_0 is pure imaginary, then the asymptotics for $\lambda_\pm(\varepsilon,t)$ are

$$\begin{aligned}\lambda_\pm(\varepsilon,t)=&\lambda_0\pm\frac{\varepsilon}{4\lambda_0}\sqrt{4t^2n^2-\lambda_0^2\alpha_0^2(n)}\\&+\frac{\varepsilon^2}{4\lambda_0}\left(\left(2-\frac{n^2}{2\lambda_0^2}\right)t^2-\frac{7\alpha_0^2(n)}{8}+\lambda_0^2\alpha_1(n)\right)+O(\varepsilon^3),\end{aligned} \tag{2.18}$$

where α_0, α_1 are defined by (2.16).

If $\frac{1}{2}-C\varepsilon\leqslant|\tau|\leqslant\frac{1}{2}$, where C is from (2.12), then operator pencil $\mathcal{H}_\varepsilon^\tau$ has exactly two eigenvalues $\lambda_\pm(\varepsilon,t)$, $|\tau|=\frac{1}{2}+\varepsilon t$ converging to $\lambda_0:=\lambda^{(n)}(0,\frac{1}{2})$ uniformly in t. These eigenvalues have the asymptotics

$$\lambda_\pm(\varepsilon,t)=\lambda_0-\frac{\varepsilon t}{2\lambda_0}\pm\frac{\varepsilon}{4\lambda_0}\sqrt{(4n+2)^2t^2+\lambda_0^2\alpha_0^2(n+1)}+O(\varepsilon^2). \tag{2.19}$$

In addition to the eigenvalues described in Theorem 2.2, operator pencil $\mathcal{H}_\varepsilon$ can also have eigenvalues converging to the essential spectrum as $\varepsilon\to+0$. Our next main result is devoted to the study of such eigenvalues. In order to formulate this result, we first introduce additional notations.

Given $n\in\mathbb{Z}$, we introduce the Jost functions Y_1, Y_2 as the solutions to the equation

$$-Y''+VY-n^2Y=0,\quad x\in\mathbb{R}, \tag{2.20}$$

behaving at infinity as

$$Y_1(x,n)\sim e^{inx},\quad x\to+\infty,\qquad Y_2(x,n)\sim e^{-inx},\quad x\to-\infty. \tag{2.21}$$

Hereinafter the symbol '$\sim$' stands for an asymptotics up to an exponentially small term $O(e^{-\delta|x|})$ with some fixed δ independent on ε. Functions Y_1, Y_2 are related by the identities [1, Ch. 2, Sec. 2.6.5, Eqs. (6.61), (6.67")]:

$$\begin{aligned} &Y_1(x,n) = -\overline{b(n)}Y_2(x,n) + a(n)\overline{Y_2(x,n)},\\ &Y_2(x,n) = a(n)\overline{Y_1(x,n)} + b(n)Y_1(x,n), \end{aligned} \tag{2.22}$$

where $a(n)$, $b(n)$ are the transmission and reflection coefficients, respectively, satisfying

$$|a(n)|^2 - |b(n)|^2 = 1. \tag{2.23}$$

We represent $a(n)$ as

$$a(n) = |a(n)|e^{i\theta(n)}, \quad 0 \leqslant \theta(n) < 2\pi, \tag{2.24}$$

and we denote $a_{\mathrm{r}}(n) := \operatorname{Re} a(n)$, $a_{\mathrm{i}}(n) := \operatorname{Im} a(n)$, $b_{\mathrm{r}}(n) := \operatorname{Re} b(n)$, $b_{\mathrm{i}}(n) := \operatorname{Re} b(n)$.

We denote

$$\begin{aligned}
&\kappa(\lambda) = \sqrt{\lambda^2 - \varkappa_*}, \quad \rho_0(\lambda) := \int\limits_0^{2\pi} \gamma(x)\frac{\sin 2\kappa(\lambda)(\pi - t)}{2\kappa(\lambda)}\,dt,\\
&\hat\rho := (n^2 - \varkappa_*)\rho_0^2(\lambda_0) - \left(n + \frac{\varkappa_*}{n}\right)\int\limits_0^{2\pi} dx\,\gamma(x)\int\limits_0^{x}\gamma(t)(\pi + t - x)\sin 2n(t-x)\,dt,\\
&\rho_{12}^{(2)}(\lambda_0) := -\int\limits_0^{2\pi} dx\,\gamma(x)\frac{\sin nx}{n}\int\limits_0^{x}\gamma(t)\frac{\sin nt}{n}\frac{\sin n(x-t)}{n}\,dt,\\
&\rho_{21}^{(2)}(\lambda_0) := \int\limits_0^{2\pi} dx\,\gamma(x)\cos nx\int\limits_0^{x}\gamma(t)\cos nt\,\frac{\sin n(x-t)}{n}\,dt,\\
&\zeta_\pm = -\frac{1}{2}\left(\theta(n) \pm \arccos\frac{b_{\mathrm{r}}(n)}{|a(n)|}\right),\\
&X_\pm(x) := 2\operatorname{Re} e^{i\zeta_\pm}Y_1(x,n),
\end{aligned} \tag{2.25}$$

where the branch of the square root in the definition of $\kappa(\lambda)$ is fixed by the condition $\sqrt{1} = 1$. Assume that $|a_{\mathrm{i}}(n)| \neq |b_{\mathrm{r}}(n)|$ and let

$$\begin{aligned}
&\Upsilon_\pm(n) := \frac{|a(n)|\sin(\zeta_\pm + \theta(n)) + b_{\mathrm{i}}(n)}{4n|a(n)|\sin(2\zeta_\pm + \theta(n))},\\
&\hat\zeta_\pm = i\lambda_0\Upsilon_\pm(n)\lim_{N\to+\infty}\int\limits_{-2\pi N}^{2\pi N}\left(\gamma - \frac{n\rho_0(\lambda_0)\sin 2\zeta_\pm}{\pi}\right)X_\pm^2\,dx + i\Upsilon_\pm(n)S_1^\pm,
\end{aligned} \tag{2.26}$$

$$\begin{aligned}
S_1^\pm :=& \frac{1}{\pi n\lambda_0\rho_0(\lambda_0)\cos^2 2\zeta_\pm}\Bigg(\frac{\varpi_\pm\lambda_0\hat\rho}{\pi}\sin 2\zeta_\pm - \lambda_0^2\rho_0^2(\lambda_0)(\varpi_\pm^2 - 1)\sin^2 2\zeta_\pm\cos 2\zeta_\pm\\
&- 2n\rho_0(\lambda_0)\left(n\rho_0(\lambda_0) + \lambda_0^2\frac{d\rho_0}{d\kappa}(\lambda_0)\right)\left(\varpi_\pm + \frac{\varpi_\pm^2 + 1}{2}\sin 2\zeta_\pm\right)\sin 2\zeta_\pm\\
&- 2\pi n^2\lambda_0^2\rho_{12}^{(2)}(\lambda_0)(\varpi_\pm\cos\zeta_\pm + \sin\zeta_\pm)^2 - 2\pi\lambda_0^2\rho_{21}^{(2)}(\lambda_0)(\varpi_\pm\sin\zeta_\pm + \cos\zeta_\pm)^2\Bigg),
\end{aligned}$$

$$\varpi_\pm := -\operatorname{Im}\left(a(n)e^{-2i\zeta_\pm} - b(n)\right) = |a(n)|\sin(2\zeta_\pm - \theta(n)) + b_{\mathrm{i}}(n).$$

If $a_{\mathrm{i}}(n) = \mp b_{\mathrm{r}}(n)$, we let

$$\hat\xi_\pm = -\frac{a_{\mathrm{r}} \pm b_{\mathrm{i}}}{4nb_{\mathrm{i}}}\left(\lambda_0\lim_{\substack{N\to+\infty\\ N\in\mathbb{N}}}\int\limits_{-2\pi N}^{2\pi N}\left(\gamma(x) - \frac{n\rho_0(\lambda_0)}{\pi}\right)X_\pm^2(x)\,dx - 4n(1 + (a_{\mathrm{r}} \pm b_{\mathrm{i}})^2)S_2\right),$$

$$S_2 := -\frac{1}{2\pi n\lambda_0\rho_0(\lambda_0)}\left((n^2+\lambda_0^2)\rho_0^2(\lambda_0)+n\lambda_0^2\rho_0(\lambda_0)\frac{d\rho_0}{d\kappa}(n)-2\pi n^2\lambda_0^2\rho_{12}^{(2)}(\lambda_0)\right).$$

We also observe that as $a_{\mathrm{i}}(n)=\mp b_{\mathrm{r}}(n)$, then $\zeta_+=\frac{\pi}{4}$, $\zeta_-=\frac{3\pi}{4}$.

Now we are in position to formulate our next main result.

Theorem 2.6. *The following statements hold true.*

1) All the isolated eigenvalues of operator pencil $\mathcal{H}_\varepsilon$ converging to the essential spectrum converge $\pm\sqrt{n^2+\varkappa_*}$, $n\in\mathbb{Z}$.

2) Let $\lambda_0 := n\sqrt{1+\frac{\varkappa_*}{n^2}}\neq 0$, $\rho_0(\lambda_0)\neq 0$. All the isolated eigenvalues of operator pencil $\mathcal{H}_\varepsilon$ converging to λ_0 as $\varepsilon\to+0$ satisfy the asymptotics

$$\lambda_\varepsilon=\lambda_0+\frac{i\varepsilon n\rho_0(\lambda_0)}{2\pi}\sin 2\zeta_\pm+\varepsilon^2\Lambda+O(\varepsilon^2) \tag{2.27}$$

for one of $\zeta_\pm$ with some constant Λ.

3) Let $\lambda_0 := n\sqrt{1+\frac{\varkappa_*}{n^2}}\neq 0$, $\rho_0(\lambda_0)\neq 0$, $|a_{\mathrm{i}}(n)|\neq|b_{\mathrm{r}}(n)|$ and choose one of numbers ζ_+ or ζ_-. Under the condition

$$\begin{aligned}
&\rho_0(\lambda_0)\cos 2\zeta_\pm \operatorname{Im}\lambda_0<0 && \text{if } \lambda_0 \text{ is pure imaginary},\\
&\frac{\hat\rho}{2\pi}\tan\left(\theta(n)\pm\arccos\frac{b_{\mathrm{r}}(n)}{|a(n)|}\right)<0 && \text{if } \lambda_0 \text{ is real},
\end{aligned} \tag{2.28}$$

operator pencil $\mathcal{H}_\varepsilon$ has the unique isolated eigenvalue converging to λ_0 with asymptotics (2.27). This eigenvalue is simple and constant Λ in (2.27) is given by the formula

$$\Lambda=-\frac{in\rho_0(\lambda_0)\hat\zeta_\pm\cos 2\zeta_\pm}{\pi}-\frac{\varkappa_*\rho_0^2(\lambda_0)\sin^2 2\zeta_\pm}{8\pi^2\lambda_0}. \tag{2.29}$$

Under the condition

$$\begin{aligned}
&\rho_0(\lambda_0)\cos 2\zeta_\pm \operatorname{Im}\lambda_0>0 && \text{if } \lambda_0 \text{ is pure imaginary},\\
&\frac{\hat\rho}{2\pi}\tan\left(\theta(n)\pm\arccos\frac{b_{\mathrm{r}}(n)}{|a(n)|}\right)>0 && \text{if } \lambda_0 \text{ is real},
\end{aligned} \tag{2.30}$$

operator pencil $\mathcal{H}_\varepsilon$ has no isolated eigenvalues with asymptotics (2.27).

4) Let $\lambda_0 := n\sqrt{1+\frac{\varkappa_*}{n^2}}\neq 0$, $\rho_0(\lambda_0)\neq 0$, $a_{\mathrm{i}}(n)=\mp b_{\mathrm{r}}(n)$, $b_{\mathrm{i}}(n)\neq 0$ and choose one of numbers ζ_+ or ζ_-. Under the condition

$$\operatorname{Re}\frac{\lambda_0\rho_0(\lambda_0)b_{\mathrm{i}}}{a_{\mathrm{r}}}\left(\hat\xi_\pm-\frac{\hat\rho}{\pi\lambda_0\rho_0(\lambda_0)}\right)<0, \tag{2.31}$$

operator pencil $\mathcal{H}_\varepsilon$ has the unique isolated eigenvalue converging to λ_0 with asymptotics (2.27). This eigenvalue is simple and $\sin 2\zeta_\pm=\pm1$,

$$\Lambda=-\frac{\varkappa_*\rho_0^2(\lambda_0)+4n\hat\rho}{8\pi^2\lambda_0} \tag{2.32}$$

in (2.27). Under the condition

$$\operatorname{Re}\frac{\lambda_0\rho_0(\lambda_0)b_{\mathrm{i}}}{a_{\mathrm{r}}}\left(\hat\xi_\pm-\frac{\hat\rho}{\pi\lambda_0\rho_0(\lambda_0)}\right)>0, \tag{2.33}$$

operator pencil $\mathcal{H}_\varepsilon$ has no isolated eigenvalues with asymptotics (2.27).

Let us discuss briefly the main results of the work. Theorem 2.1 states the convergence of spectrum $\sigma(\mathcal{H}_\varepsilon)$ to the spectrum of the limiting operator pencil $\mathcal{H}_0+\varkappa_*-\lambda^2$. Relation (2.5) provides explicitly a domain in which spectrum $\sigma(\mathcal{H}_\varepsilon)$ is located.

Theorem 2.2 presents the results on perturbation of the isolated eigenvalues of the limiting operator pencil. While series (2.6), (2.7) are very natural and expectable, the

situation with the zero limiting eigenvalue is quite non-trivial. As we see, the zero limiting eigenvalue is always an eigenvalue of $\mathcal{H}_\varepsilon$. And there can be an additional eigenvalue converging to zero. The existence of such eigenvalue is controlled by conditions (2.8), (2.9). In other words, zero limiting eigenvalue can generate either one or two perturbed eigenvalues.

The essential spectrum is described by Theorem 2.4. It has a band structure, see (2.11). Outside small neighbourhoods of points $\pm\sqrt{n^2+\varkappa_*}$ and $\pm\sqrt{\left(n-\frac{1}{2}\right)^2+\varkappa_*}$, all the bands are smooth curves, whose structure is approximately described by identity (2.13). In the vicinities of the above mentioned points we see the bifurcation of the bands. Namely, in the vicinity of points $\lambda_0=\pm\sqrt{n^2+\varkappa_*}$, the bands of the essential spectrum are approximately described by the curves

$$\begin{aligned}\lambda_\pm^{(n)}(\varepsilon,t)\approx&\lambda_0\pm\frac{i\varepsilon}{4\lambda_0}\sqrt{\lambda_0^2\alpha_0^2(n)-4t^2n^2}\\&+\frac{\varepsilon^2}{4\lambda_0}\left(\left(2-\frac{n^2}{2\lambda_0^2}\right)t^2-\frac{7\alpha_0^2(n)}{8}+\lambda_0^2\alpha_1(n)\right).\end{aligned}$$

These are parabolas bifurcating into a segment, no matter whether λ_0 is real or pure imaginary, see Figure 1. In the vicinities of the points $\lambda_0=\pm\sqrt{\left(n-\frac{1}{2}\right)^2+\varkappa_*}$ the structure is due to asymptotics (2.19). The right hand side of this formula is real for real λ_0. For pure imaginary λ_0 we again deal with parabolas.

The last theorem, Theorem 2.6, is devoted to the isolated eigenvalues converging to the essential spectrum. As it states, such eigenvalues can exist only in the vicinities of points $\lambda_0=\pm\sqrt{n^2+\varkappa_*}$. We observe that near points λ_0 with $n\neq 0$ essential spectrum has a form of parabolas. Our result concerns the isolated eigenvalues near points λ_0 with $n\neq 0$. All possible eigenvalues can have only certain asymptotic behaviour, see (2.27). Constants $\zeta_\pm$ are determined by (2.25) via transmission and reflection coefficients $a(n)$ and $b(n)$. To each of constants ζ_+ and ζ_-, at most one simple isolated eigenvalue can be associated. The existence or absence of such eigenvalue is controlled by conditions (2.28), (2.30), (2.31), (2.33). For each λ_0, operator pencil $\mathcal{H}_\varepsilon$ can have none or one or two isolated eigenvalues converging to λ_0 as $\varepsilon\to+0$. A possible location of the discussed isolated eigenvalues is demonstrated in Figure 1. We also note that the above results are proved under the assumptions $\lambda_0\neq 0$, $\rho_0(\lambda_0)\neq 0$ and $b_{\mathrm{i}}(n)\neq 0$ if $a_{\mathrm{i}}(n)=\mp b_{\mathrm{r}}(n)$. Without these assumptions our technique does not work and at least, one should modify it to obtain similar results.

3 Convergence of spectrum

In this section we prove Theorem 2.1. We begin with the absence of the residual spectrum. If $\lambda\in\sigma_{\mathrm{res}}(\mathcal{H}_\varepsilon(\lambda))$, in accordance with the definition, there is no bounded inverse operator for $\mathcal{H}_\varepsilon(\lambda)$ in $L_2(\mathbb{R})$ and there exists a fixed constant $C(\varepsilon,\lambda)>0$ such that

$$\|\mathcal{H}_\varepsilon(\lambda)f\|_{L_2(\mathbb{R})}\geqslant C\|f\|_{L_2(\mathbb{R})}$$

for all $f\in L_2(\mathbb{R})$. Indeed, if this inequality is false, then we arrive either at the definition of the essential spectrum or at the existence of an eigenfunction associated with λ. Therefore, the image of $\mathcal{H}_\varepsilon(\lambda)$ does not coincide with $L_2(\mathbb{R})$; we also note that the image of $\mathcal{H}_\varepsilon(\lambda)$ is a subspace in $L_2(\mathbb{R})$. Hence, the kernel of the adjoint operator $(\mathcal{H}_\varepsilon(\lambda))^*$ is non-empty and there exists a function $\psi\in W_2^2(\mathbb{R})$ such that $(\mathcal{H}_\varepsilon(\lambda))^*\psi=0$. By (2.2) it yields that $-\overline{\lambda}$ is an eigenvalue of operator pencil $\mathcal{H}_\varepsilon(\lambda)$ and the associated eigenfunction is $\mathcal{P}\psi$. Therefore, by (2.3), $\lambda\in\sigma_{\mathrm{pnt}}(\mathcal{H}_\varepsilon)$ that contradicts the definition of the residual spectrum. Hence, $\sigma_{\mathrm{res}}(\mathcal{H}_\varepsilon)$ is empty.

We proceed to the proof of (2.5). Since operator $\mathcal{H}_0$ is self-adjoint, we have

$$\|(\mathcal{H}_0+\varkappa_*-\lambda^2)^{-1}\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})}=\frac{1}{\operatorname{dist}(\lambda^2-\varkappa_*^2,\sigma(\mathcal{H}_0))},\tag{3.1}$$

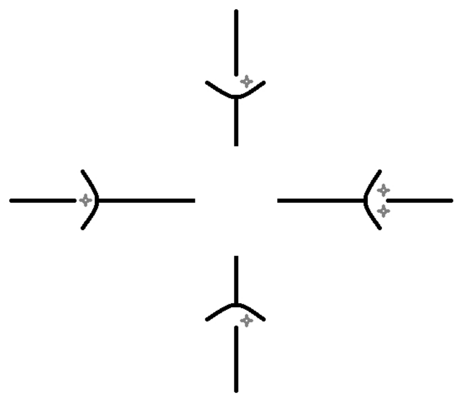

Figure 1: Approximate shape of horizontal and vertical bands in the essential spectrum and possible isolated eigenvalues converging to the essential spectrum

where $\|\cdot\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})}$ stands for the norm of a bounded operator in $L_2(\mathbb{R})$. Since

$$\big(\mathcal{H}_\varepsilon(\lambda)\big)^{-1}=(\mathcal{H}_0+\varkappa_*+i\varepsilon\lambda\gamma-\lambda^2)^{-1}=(\mathcal{H}_0+\varkappa_*-\lambda^2)^{-1}\Big(I+i\varepsilon\lambda\gamma(\mathcal{H}_0+\varkappa_*-\lambda^2)^{-1}\Big)^{-1},$$

the inverse operator for $\mathcal{H}_\varepsilon(\lambda)$ is well-defined as a bounded operator in $L_2(\mathbb{R})$ provided

$$\varepsilon|\lambda|\|\gamma(\mathcal{H}_0+\varkappa_*-\lambda^2)^{-1}\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})}<1.$$

In view of (3.1), this inequality holds true if

$$\frac{\varepsilon|\lambda|\sup\limits_{[-\pi,\pi]}|\gamma|}{\operatorname{dist}(\lambda^2-\varkappa_*,\sigma(\mathcal{H}_0))}<1, \tag{3.2}$$

i.e., such λ are in the resolvent set of operator pencil $\mathcal{H}_\varepsilon(\lambda)$. The opposite inequality for (3.2) leads us to (2.5).

4 Perturbation of limiting isolated eigenvalues

This section is devoted to the proof of Theorem 2.2. Let $\varkappa=\varkappa_j$ and $\Psi=\Psi_j$ be an eigenvalue of operator $\mathcal{H}_0$ and the associated real-valued eigenfunction normalized in $L_2(\mathbb{R})$. To describe the points in the spectrum of operator pencil $\mathcal{H}_\varepsilon(\lambda)$ converging to $\pm\sqrt{\varkappa+\varkappa_*}$ as $\varepsilon\to+0$, we shall study the singularities of $(\mathcal{H}_\varepsilon(\lambda))^{-1}$ as λ is close to $\pm\sqrt{\varkappa+\varkappa_*}$.

Consider the equation $\mathcal{H}_\varepsilon(\lambda)u=f$, which we rewrite as

$$(\mathcal{H}_0+\varkappa_*-\lambda^2+i\varepsilon\lambda\gamma)u=f. \tag{4.1}$$

For λ^2 close to $\varkappa$, the resolvent of operator $\mathcal{H}_0$ satisfies the representation

$$(\mathcal{H}_0+\varkappa_*-\lambda^2)^{-1}f=\frac{(f,\Psi)_{L_2(\mathbb{R})}}{\varkappa_*+\varkappa-\lambda^2}\Psi+\mathcal{A}_1(\lambda)f, \tag{4.2}$$

where f is an arbitrary function, $\mathcal{A}_1(z):L_2(\mathbb{R})\to W_2^2(\mathbb{R})$ is a bounded operator holomorphic w.r.t. z close to $\varkappa_*+\varkappa$.

We apply the resolvent $(\mathcal{H}_0+\varkappa_*-\lambda^2)^{-1}$ to equation (4.1) and employ representation (4.2):

$$u+\frac{i\varepsilon\lambda(\gamma u,\Psi)_{L_2(\mathbb{R})}}{\varkappa_*+\varkappa-\lambda^2}\Psi+i\varepsilon\lambda\mathcal{A}_1(\lambda^2)\gamma u=\frac{(f,\Psi)_{L_2(\mathbb{R})}}{\varkappa_*+\varkappa-\lambda^2}\Psi+\mathcal{A}_1(\lambda^2)f. \tag{4.3}$$

Since we consider λ in the vicinity of points $\pm\sqrt{\varkappa+\varkappa_*}$, it is bounded and therefore, operator $i\varepsilon\lambda\mathcal{A}_1(\lambda^2)$ is small for sufficiently small ε. Hence, operator

$$\mathcal{A}_2(\varepsilon,\lambda) := \big(I + i\varepsilon\lambda\mathcal{A}_1(\lambda^2)\big)^{-1} \tag{4.4}$$

is well-defined as a bounded operator from $L_2(\mathbb{R})$ into $W_2^2(\mathbb{R})$. It is obvious that this operator is jointly holomorphic w.r.t. small ε and λ close to $\pm\sqrt{\varkappa+\varkappa_*}$. We apply operator $\mathcal{A}_2$ to equation (4.3) and get:

$$u + \frac{i\varepsilon\lambda(\gamma u,\Psi)_{L_2(\mathbb{R})}}{\varkappa_*+\varkappa-\lambda^2}\mathcal{A}_2(\varepsilon,\lambda)\Psi = \frac{(f,\Psi)_{L_2(\mathbb{R})}}{\varkappa_*+\varkappa-\lambda^2}\mathcal{A}_2(\varepsilon,\lambda)\Psi + \mathcal{A}_2(\varepsilon,\lambda)\mathcal{A}_1(\lambda^2)f. \tag{4.5}$$

We calculate the scalar product of this equation with $\gamma\Psi$ in $L_2(\mathbb{R})$ and we solve the obtained equation w.r.t. $(\gamma u,\Psi)_{L_2(\mathbb{R})}$:

$$(\gamma u,\Psi)_{L_2(\mathbb{R})} = \frac{(f,\Psi)_{L_2(\mathbb{R})}(\gamma\mathcal{A}_2(\varepsilon,\lambda)\Psi,\Psi)_{L_2(\mathbb{R})} + (\varkappa_*+\varkappa-\lambda^2)(\gamma\mathcal{A}_2(\varepsilon,\lambda)\Psi,\Psi)_{L_2(\mathbb{R})}}{\varkappa_*+\varkappa-\lambda^2+i\varepsilon\lambda(\gamma\mathcal{A}_2(\varepsilon,\lambda)\Psi,\Psi)_{L_2(\mathbb{R})}}.$$

It allows us to solve equation (4.5):

$$\big(\mathcal{H}_\varepsilon(\lambda)\big)^{-1}f = u = \frac{\Big(\big(I - i\varepsilon\lambda\gamma\mathcal{A}_2(\varepsilon,\lambda)\mathcal{A}_1(\lambda^2)\big)f,\Psi\Big)_{L_2(\mathbb{R})}}{\varkappa_*+\varkappa-\lambda^2+i\varepsilon\lambda(\gamma\mathcal{A}_2(\varepsilon,\lambda)\Psi,\Psi)_{L_2(\mathbb{R})}} + \mathcal{A}_2(\varepsilon,\lambda)\mathcal{A}_1(\lambda^2)f.$$

Thus, operator $\big(\mathcal{H}_\varepsilon(\lambda)\big)^{-1}$ is well-defined as a bounded operator in $L_2(\mathbb{R})$ provided the denominator in the first term in the right hand side is non-zero. Therefore, the points of the spectrum of $\mathcal{H}_\varepsilon(\lambda)$ in the vicinity of $\pm\sqrt{\varkappa+\varkappa_*}$ are determined by the equation

$$\varkappa_*+\varkappa-\lambda^2+i\varepsilon\lambda(\gamma\mathcal{A}_2(\varepsilon,\lambda)\Psi,\Psi)_{L_2(\mathbb{R})} = 0. \tag{4.6}$$

Each root to this equation is a simple eigenvalue of operator pencil $\mathcal{H}_\varepsilon(\lambda)$. As one can check by equation (4.5) with $f=0$, we can choose the associated eigenfunction as

$$\psi_\varepsilon = \frac{\mathcal{A}_2(\varepsilon,\lambda)\Psi}{(\mathcal{A}_2(\varepsilon,\lambda)\Psi,\Psi)_{L_2(\mathbb{R})}}. \tag{4.7}$$

Thanks to the definition of operator $\mathcal{A}_2(\varepsilon,\lambda)$, the denominator is non-zero for sufficiently small ε and the identity

$$(\psi_\varepsilon,\Psi)_{L_2(\mathbb{R})} = 1 \tag{4.8}$$

holds true.

Let us study the solvability of equation (4.6). We first observe that the function

$$h(\varepsilon,\lambda) := \big(\gamma\mathcal{A}_2(\varepsilon,\lambda)\Psi,\Psi\big)_{L_2(\mathbb{R})} \tag{4.9}$$

is jointly holomorphic w.r.t. small ε and λ close to $\pm\sqrt{\varkappa_*+\varkappa}$.

Assume first that $\varkappa_*+\varkappa=0$. Then equation (4.6) has one obvious root $\lambda=0$ and is reduced to the equation

$$\lambda = i\varepsilon h(\varepsilon,\lambda). \tag{4.10}$$

And since function h is jointly holomorphic w.r.t. ε and λ, by the inverse function theorem equation (4.10) has the unique root λ_ε converging to zero as $\varepsilon\to+0$ and this root is holomorphic w.r.t. ε. Eigenvalue λ_ε is simple and in view of identities (2.3), $\overline{\lambda_\varepsilon}$ is also an eigenvalue. This is possible only if λ_ε is a pure imaginary eigenvalue.

If identity (2.8) holds true, then equation (4.10) has the only solution $\lambda_\varepsilon=0$. Indeed, it follows from definitions (4.4), (4.9) of operator $\mathcal{A}_2$ and function h that

$$\begin{aligned}&\mathcal{A}_2(\varepsilon,\lambda) = I - i\varepsilon\lambda\gamma\mathcal{A}_1(\lambda^2)\mathcal{A}_2(\varepsilon,\lambda),\\ &h(\varepsilon,\lambda) = (\gamma\Psi,\Psi)_{L_2(\mathbb{R})} - i\varepsilon\lambda h_1(\varepsilon,\lambda),\end{aligned}$$

$$h_1(\varepsilon,\lambda) := (\gamma\mathcal{A}_1(\lambda^2)\mathcal{A}_2(\varepsilon,\lambda)\Psi,\Psi)_{L_2(\mathbb{R})}.$$

Hence, under condition (2.8) equation (4.10) casts into the form: $\lambda = -\varepsilon^2\lambda h_1(\varepsilon,\lambda)$ and it has the only root $\lambda_\varepsilon = 0$. The associated eigenfunction is still Ψ.

If $\varkappa + \varkappa_* \neq 0$, equation (4.6) is equivalent to the pair of equations

$$\lambda = \pm\sqrt{\varkappa_* + \varkappa + i\varepsilon h(\varepsilon,\lambda)}.$$

And again by the inverse function theorem each of the above equations has the unique (simple) root converging to zero as $\varepsilon \to +0$ and this root is holomorphic w.r.t. ε. A pair of these roots being eigenvalues of operator pencil $\mathcal{H}_\varepsilon(\lambda)$ due to (2.2). We denote the eigenvalue converging to $\sqrt{\varkappa_j + \varkappa_*}$ by λ_ε, while the other eigenvalue is $\overline{\lambda_\varepsilon}$.

Since the eigenfunctions associated with the above roots of (4.6) are given by (4.7) and operator $\mathcal{A}_2(\varepsilon,\lambda)$ is jointly holomorphic w.r.t. ε and λ, these eigenfunctions are holomorphic w.r.t. ε in the norm of $W_2^2(\mathbb{R})$.

Let ψ_ε be the eigenfunction defined by (4.7) and associated with eigenvalue λ_ε. They are represented by the convergent series

$$\lambda_\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n \Lambda_n, \quad \psi_\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n \psi_n, \quad \psi_0 := \Psi,$$

where the latter series converges in $W_2^2(\mathbb{R})$. We substitute these series into eigenvalue equation $\mathcal{H}_\varepsilon(\lambda_\varepsilon)\psi_\varepsilon = 0$ and condition (4.8) and equate the coefficients at the like powers of ε. It implies the equations for ψ_n

$$(\mathcal{H}_0 - \varkappa)\psi_n = -i\gamma \sum_{m=0}^{n-1} \Lambda_m \psi_{n-m-1} + \sum_{k=0}^{n-1}\sum_{m=0}^{n-k} \Lambda_m \Lambda_{n-m-k}\psi_k \tag{4.11}$$

and the conditions

$$(\psi_n, \Psi)_{L_2(\mathbb{R})} = 0, \quad n \geqslant 1. \tag{4.12}$$

The solvability condition is the orthogonality of the right hand side in (4.11) to Ψ in $L_2(\mathbb{R})$. It determines Λ_n:

$$\Lambda_n = -\frac{1}{2\Lambda_0}\sum_{m=1}^{n-1} \Lambda_m \Lambda_{n-m} + \frac{i}{2\Lambda_0}\sum_{m=0}^{n-1} \Lambda_m (\gamma\psi_{n-m-1}, \Psi)_{L_2(\mathbb{R})}. \tag{4.13}$$

It is also clear that under the solvability condition there exists the unique solution to (4.11) satisfying (4.12).

As $n = 1, 2$, formula (4.13) imply (2.7).

We proceed to the case $\varkappa_* + \varkappa = 0$. In this case zero is an eigenvalue and the associated eigenfunction is Ψ. If condition (2.9) holds true, there is one more eigenvalue λ_ε converging to zero as $\varepsilon \to 0$.

Eigenvalue λ_ε and the associated eigenfunction are represented by the convergent series

$$\lambda_\varepsilon = \sum_{n=1}^{\infty} \varepsilon^n \Lambda_n, \quad \psi_\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n \psi_n, \quad \psi_0 := \Psi.$$

The equations for functions ψ_n read as

$$(\mathcal{H}_0 - \varkappa)\psi_n = -i\gamma \sum_{m=1}^{n-1} \Lambda_m \psi_{n-m-1} + \sum_{k=0}^{n-2}\sum_{m=0}^{n-k-1} \Lambda_m \Lambda_{n-m-k}\psi_k. \tag{4.14}$$

Conditions (4.12) are to be satisfied as well.

As $n = 1$, the right hand side in (4.14) vanishes and therefore $\psi_1 = 0$.

We could find coefficients Λ_n by means of the solvability conditions of equations (4.14). But a more elegant way is to multiply equation $\mathcal{H}_\varepsilon(\lambda_\varepsilon)\psi_\varepsilon = 0$ by Ψ and integrate twice by parts. It implies

$$\lambda_\varepsilon = i\varepsilon(\gamma\psi_\varepsilon, \Psi)_{L_2(\mathbb{R})} = i\sum_{n=1}^{\infty} \varepsilon^n (\gamma\psi_{n-1}, \Psi)_{L_2(\mathbb{R})}, \quad \Lambda_n = i(\gamma\psi_{n-1}, \Psi)_{L_2(\mathbb{R})}.$$

In particular, since $\psi_1 = 0$, it implies that $\Lambda_2 = 0$.

5 Essential spectrum

In this section we study the essential spectrum of operator pencil $\mathcal{H}_\varepsilon$. We begin with proving Lemma 2.3.

5.1 Proof of Lemma 2.3.

In the same way as in the proof of Theorem 2.1 we can check easily that the residual spectrum of operator pencil $\mathcal{H}_\varepsilon^\tau$ is empty for each fixed ε and τ. The essential spectrum of $\mathcal{H}_\varepsilon^\tau$ is also empty. Indeed, if $\lambda \in \sigma_{\mathrm{ess}}(\mathcal{H}_\varepsilon^\tau)$ and u_n is an associated characteristic sequence, then

$$\|iu_n' + \tau u_n\|^2_{L_2(-\pi,\pi)} + i\varepsilon\lambda(\gamma u_n, u_n)_{L_2(-\pi,\pi)} - \lambda^2\|u_n\|^2_{L_2(-\pi,\pi)} = 0.$$

Thus, u_n is also bounded in $W_2^1(-\pi,\pi)$ and by the compact embedding of $W_2^1(-\pi,\pi)$ into $L_2(-\pi,\pi)$ sequence u_n is compact in $L_2(-\pi,\pi)$. This contradicts to the definition of the essential spectrum. Hence, $\sigma(\mathcal{H}_\varepsilon^\tau) = \sigma_{\mathrm{pnt}}(\mathcal{H}_\varepsilon^\tau)$.

5.2 Proof of Theorem 2.4.

We introduce an auxiliary operator pencil:

$$\mathcal{H}_\varepsilon^\flat(\lambda) := -\frac{d^2}{dx^2} + i\varepsilon\lambda\gamma + \varkappa_* - \lambda^2,$$

which is operator pencil $\mathcal{H}_\varepsilon(\lambda)$ in the case $V \equiv 0$. Let us show that

$$\sigma_{\mathrm{ess}}(\mathcal{H}_\varepsilon) = \sigma_{\mathrm{ess}}(\mathcal{H}_\varepsilon^\flat) = \sigma(\mathcal{H}_\varepsilon^\flat). \tag{5.1}$$

The proof of the first identity reproduces literally the proof of Lemma 3.3 in [2] and of Lemma 2.3 in [3]. To prove the second identity, we observe that $\mathcal{H}_\varepsilon^\flat(\lambda)$ is a periodic operator. Then we can employ Floquet-Bloch theory to see that operator $\mathcal{H}_\varepsilon^\flat(\lambda)$ has a bounded inverse in $L_2(\mathbb{R})$ provided λ is not in the spectrum of $\mathcal{H}_\varepsilon^\tau$, or, thanks to Lemma 2.3, λ is not an eigenvalue of operator pencil $\mathcal{H}_\varepsilon^\tau$ for some $\tau \in [-\frac{1}{2}, \frac{1}{2})$. Hence,

$$\sigma(\mathcal{H}_\varepsilon^\flat) = \bigcup_{\tau\in[-\frac{1}{2},\frac{1}{2})} \sigma(\mathcal{H}_\varepsilon^\tau). \tag{5.2}$$

Due to (5.2) and Lemma 2.3, for each $\lambda \in \sigma(\mathcal{H}_\varepsilon^\flat)$ there exists $\tau \in [-\frac{1}{2}, \frac{1}{2})$ and a function $\psi \in W_2^2(-\pi,\pi)$ satisfying periodic boundary conditions such that λ and ψ are an eigenvalue and an associated eigenfunction of $\mathcal{H}_\varepsilon^\tau$. We continue function ψ periodically on the whole axis. Multiplying then function $e^{i\tau x}\psi(x)$ by an appropriate sequence of cut-off functions, it is easy to construct a characteristic sequence for $\mathcal{H}_\varepsilon^\flat$ associated with λ. It proves the second identity in (5.1).

Since operator pencil $\mathcal{H}_\varepsilon^\flat$ does not involve potential V, the spectrum of $\mathcal{H}_\varepsilon^\flat$ satisfy both identities (2.3) and (2.4). Hence, the essential spectrum of $\mathcal{H}_\varepsilon$ is symmetric w.r.t. the real and imaginary axes. A similar fact is obviously true for the point spectrum of $\mathcal{H}_\varepsilon^\tau$: if λ is an eigenvalue of $\mathcal{H}_\varepsilon^\tau$ and $\psi(x)$ is an associated eigenfunction, then $\overline{\lambda}$ is also

an eigenvalue of $\mathcal{H}_\varepsilon^\tau(\lambda)$ with the associated eigenfunction $\overline{\psi(-x)}$, while $-\overline{\lambda}$, $\overline{\psi(x)}$ and $-\lambda$, $-\psi(-x)$ are eigenvalues and the associated eigenfunctions of $\mathcal{H}_\varepsilon^{-\tau}$.

We proceed to studying the asymptotic behavior of the essential spectrum. We fix τ such that $0<|\tau|<\frac{1}{2}$. Let λ_0 be one of the eigenvalues $\pm\sqrt{(n-\tau)^2+\varkappa_*}$ for some $n\in\mathbb{Z}$. The associated eigenfunction is chosen to be orthonormalized in $L_2(-\pi,\pi)$: $\psi_0(x):=\frac{1}{\sqrt{2\pi}}e^{inx}$.

To study the eigenvalue equation for $\mathcal{H}_\varepsilon^\tau$

$$\big(\mathcal{H}_0(\tau)+\varkappa_*+i\varepsilon\lambda\gamma-\lambda^2\big)\psi_\varepsilon=0, \tag{5.3}$$

we employ the same approach as in the proof of Theorem 2.2.

The resolvent of $\mathcal{H}_0(\tau)$ satisfies the representation

$$\big(\mathcal{H}_0(\tau)+\varkappa_*-\lambda^2\big)^{-1}f=\frac{\psi_0(f,\psi_0)_{L_2(-\pi,\pi)}}{\lambda_0^2-\lambda^2}+\mathcal{A}_3(\lambda^2,\tau)f, \tag{5.4}$$

where $\mathcal{A}_3(z,\tau)$ is a bounded operator from $L_2(-\pi,\pi)$ into $W_2^2(-\pi,\pi)$. Denote $Q_\tau:=\{z\in\mathbb{C}:\ \mu_-(\tau)<|\operatorname{Re}z|<\mu_+(\tau),\ |\operatorname{Im}z|<\delta\}$, where δ is a fixed positive constant. If $n=0$, we let $\mu_-(\tau):=-\delta$, $\mu_+(\tau):=(1-|\tau|)^2$. If $n\neq 0$, then $\mu_-(\tau):=(n-1+\tau)^2$, $\mu_+(\tau):=(n+\tau)^2$ provided n and τ are of the same sign, and $\mu_-(\tau):=(n+\tau)^2$, $\mu_+(\tau):=(n+1+\tau)^2$ provided the signs of n and τ are opposite. Operator $\mathcal{A}_3(z,\tau)$ is holomorphic w.r.t. $z\in Q_\tau$ and satisfies the estimate

$$\|\mathcal{A}_3(z,\tau)\|_{L_2(\pi,\pi)\to L_2(-\pi,\pi)}=\frac{1}{\operatorname{dist}(z,\{\mu_-(\tau),\mu_+(\tau)\})}. \tag{5.5}$$

We apply operator $\big(\mathcal{H}_0(\tau)+\varkappa_*-\lambda^2\big)^{-1}$ to equation (5.3) and in view of (5.4) we get

$$\psi_\varepsilon+\frac{i\varepsilon\lambda(\gamma\psi_\varepsilon,\psi_0)_{L_2(-\pi,\pi)}}{\lambda_0^2-\lambda^2}\psi_0+i\varepsilon\lambda\mathcal{A}_3(\lambda^2,\tau)\gamma\psi_\varepsilon=0.$$

Assume that λ ranges in the vicinity of λ_0 so that $\lambda^2\in Q_\tau$. Then it follows from (5.5) that

$$\|i\varepsilon\lambda\mathcal{A}_3(\lambda^2,\tau)\gamma\|_{L_2(-\pi,\pi)\to L_2(-\pi,\pi)}\leqslant\frac{c\varepsilon}{\operatorname{dist}(\lambda^2,\{\mu_-(\tau),\mu_+(\tau)\})},$$

where constant c is independent of λ, τ and ε. Thus, provided ε is small enough, namely, as

$$\frac{c\varepsilon}{\operatorname{dist}(\lambda^2,\{\mu_-(\tau),\mu_+(\tau)\})}<1, \tag{5.6}$$

the operator

$$\mathcal{A}_4(\lambda,\tau,\varepsilon):=\big(I+i\varepsilon\lambda\mathcal{A}_3(\lambda^2,\tau)\gamma\big)^{-1}$$

is well-defined as an operator in $L_2(-\pi,\pi)$ and is jointly holomorphic w.r.t. ε and λ for each fixed τ. In the same way how equation (4.6) was obtained, we arrive at the equation

$$\lambda_0^2-\lambda^2+i\varepsilon\lambda(\gamma\mathcal{A}_4(\lambda,\tau,\varepsilon)\psi_0,\psi_0)_{L_2(-\pi,\pi)}=0. \tag{5.7}$$

Assume that $\lambda_0\neq 0$. Then as in the proof of Theorem 2.2, one can check easily that equation (5.7) has the unique solution λ_ε converging to λ_0 as $\varepsilon\to+0$, λ_ε is a simple real eigenvalue holomorphic w.r.t. ε. The first terms of the Taylor series for λ_ε are given by (2.13), (2.14). Thanks to the above discussed symmetry properties of the eigenvalues of $\mathcal{H}_\varepsilon^\tau$, if λ_0 is real, the same is true for λ_ε.

If $\lambda_0=0$, then equation (5.7) has the only solution $\lambda=0$ since $(\gamma\psi_0,\psi_0)_{L_2(-\pi,\pi)}=0$.

Inequality (5.6) provides the restriction for τ under which all the above facts are true. In view of the definition of set Q_τ, we see that inequality (5.6) is satisfied if inequality (2.12) holds true, where C is a fixed positive constant independent of ε. This constant should be chosen large enough.

5.3 Proof of Theorem 2.5

To study the behavior of the eigenvalues of $\mathcal{H}_\varepsilon^\tau(\lambda)$ as $|\tau|\leqslant C\varepsilon$ or $\frac{1}{2}-C\varepsilon\leqslant|\tau|\leqslant\frac{1}{2}$, we need to modify slightly the approach used in the previous subsection.

We first consider the case $|\tau|\leqslant C\varepsilon$, where constant C is fixed and large enough. We rescale τ as $\tau=\varepsilon t$, $|t|\leqslant C$ and we rewrite eigenvalue equation (5.3) as

$$\big(\mathcal{H}_0(0)+\varepsilon\mathcal{A}_5(\varepsilon,t,\lambda)+\varkappa_*-\lambda^2\big)\psi_\varepsilon=0,\quad \mathcal{A}_5(\varepsilon,t,\lambda):=2it\frac{d}{dx}+\varepsilon t^2+i\lambda\gamma. \tag{5.8}$$

As we discussed above, for $\varepsilon=0$, the eigenvalues of this equation are $\lambda_0=\pm\sqrt{n^2+\varkappa_*}$, $n\in\mathbb{Z}_+$, and for $n\in\mathbb{N}$ these eigenvalues are double. We choose the associated eigenfunctions as

$$\psi_0^+(x):=\frac{1}{\sqrt{\pi}}\cos nx,\quad \psi_0^-(x):=\frac{1}{\sqrt{\pi}}\sin nx.$$

As λ^2 is close to λ_0^2, the resolvent of operator $\mathcal{H}_0(0)$ satisfies the representation:

$$\big(\mathcal{H}_0(0)+\varkappa_*-\lambda^2\big)^{-1}f=\frac{1}{\lambda_0^2-\lambda^2}\big(\psi_0^+(f,\psi_0^+)_{L_2(-\pi,\pi)}+\psi_0^-(f,\psi_0^-)_{L_2(-\pi,\pi)}\big)+\mathcal{A}_6(\lambda^2)f, \tag{5.9}$$

where operator $\mathcal{A}_6(z):L_2(-\pi,\pi)\to W_2^2(-\pi,\pi)$ is bounded and holomorphic w.r.t. z close to λ_0^2. Given $f\in L_2(-\pi,\pi)$, function $u=\mathcal{A}_6(z)f$ is orthogonal to ψ_0^+ and ψ_0^- in $L_2(-\pi,\pi)$ and it solves the equation

$$(\mathcal{H}_0(0)+\varkappa_*-z)u=f^\perp,\quad f^\perp:=f-(f,\psi_0^+)_{L_2(-\pi,\pi)}\psi_0^+-(f,\psi_0^-)_{L_2(-\pi,\pi)}\psi_0^-. \tag{5.10}$$

Suppose that λ ranges in a small fixed neighborhood of λ_0. We apply the resolvent $(\mathcal{H}_0(0)+\varkappa_*-\lambda^2)^{-1}$ to equation (5.8) and employ representation (5.9) to obtain:

$$\begin{aligned}&\psi_\varepsilon+\frac{\varepsilon}{\lambda_0^2-\lambda^2}\Big(\psi_0^+(\mathcal{A}_5(\varepsilon,t,\lambda)\psi_\varepsilon,\psi_0^+)_{L_2(-\pi,\pi)}+\psi_0^-(\mathcal{A}_5(\varepsilon,t,\lambda)\psi_\varepsilon,\psi_0^-)_{L_2(-\pi,\pi)}\Big)\\&\quad+\varepsilon\mathcal{A}_6(\lambda^2)\mathcal{A}_5(\varepsilon,t,\lambda)\psi_\varepsilon=0.\end{aligned} \tag{5.11}$$

It is clear that operator

$$\mathcal{A}_7(\varepsilon,t,\lambda):=\big(I+\varepsilon\mathcal{A}_6(\lambda^2)\mathcal{A}_5(\varepsilon,t,\lambda)\big)^{-1} \tag{5.12}$$

is well-defined as a bounded operator in $L_2(-\pi,\pi)$ for sufficiently small ε and it is jointly holomorphic w.r.t. ε, t, and λ. We apply this operator to equation (5.11):

$$\begin{aligned}&\psi_\varepsilon+\frac{\varepsilon}{\lambda_0^2-\lambda^2}(\mathcal{A}_5(\varepsilon,t,\lambda)\psi_\varepsilon,\psi_0^+)_{L_2(-\pi,\pi)}\mathcal{A}_7(\varepsilon,t,\lambda)\psi_0^+\\&\quad+\mathcal{A}_7(\varepsilon,t,\lambda)\psi_0^-(\mathcal{A}_5(\varepsilon,t,\lambda)\psi_\varepsilon,\psi_0^-)_{L_2(-\pi,\pi)}\mathcal{A}_7(\varepsilon,t,\lambda)\psi_0^-=0.\end{aligned} \tag{5.13}$$

Then we apply operator $\mathcal{A}_5$ to this equation and calculate the scalar product of the result with $\psi_0^\pm$. It leads us to a pair of linear equations:

$$\left(E+\frac{\varepsilon}{\lambda_0^2-\lambda^2}\mathcal{A}_8(\varepsilon,t,\lambda)\right)\begin{pmatrix}(\mathcal{A}_5(\varepsilon,t,\lambda)\psi_\varepsilon,\psi_0^+)_{L_2(-\pi,\pi)}\\(\mathcal{A}_5(\varepsilon,t,\lambda)\psi_\varepsilon,\psi_0^-)_{L_2(-\pi,\pi)}\end{pmatrix}=0, \tag{5.14}$$

$$\mathcal{A}_8(\varepsilon,t,\lambda):=\begin{pmatrix}\mathcal{A}_8^{(++)}(\varepsilon,t,\lambda) & \mathcal{A}_8^{(-+)}(\varepsilon,t,\lambda)\\ \mathcal{A}_8^{(+-)}(\varepsilon,t,\lambda) & \mathcal{A}_8^{(--)}(\varepsilon,t,\lambda)\end{pmatrix},$$

$$\mathcal{A}_8^{(\natural\sharp)}(\varepsilon,t,\lambda):=\big(\mathcal{A}_5(\varepsilon,t,\lambda)\mathcal{A}_7(\varepsilon,t,\lambda)\psi_0^\natural,\psi_0^\sharp\big)_{L_2(-\pi,\pi)},\quad \natural,\sharp=+,-,$$

where E is the unit matrix. Equation (5.14) has a non-trivial solution if and only if

$$\det\big((\lambda_0^2-\lambda^2)E+\varepsilon\mathcal{A}_8(\varepsilon,t,\lambda)\big)=0. \tag{5.15}$$

This is the equation determining eigenvalues of equation (5.8). The associated eigenfunctions are given by formula (5.13), where scalar products $(\mathcal{A}_5\psi_\varepsilon,\psi_0^\pm)_{L_2(-\pi,\pi)}$ are the components of the associated non-trivial solution to equation (5.14).

Equation (5.15) can be rewritten as

$$(\lambda^2-\lambda_0^2)^2-\varepsilon h(\varepsilon,t,\lambda)=0,\quad h(\varepsilon,t,\lambda):=(\lambda^2-\lambda_0^2)\operatorname{tr}\mathcal{A}_8(\varepsilon,t,\lambda)-\varepsilon\det\mathcal{A}_8(\varepsilon,t,\lambda). \tag{5.16}$$

Lemma 5.1. *Equation (5.16) has exactly two roots $\lambda_\pm(\varepsilon,t)$ in the vicinity of λ_0 counting multiplicities. These roots converge to λ_0 as $\varepsilon\to+0$ uniformly in t and they are jointly continuous in ε and t.*

Proof. In the vicinity of λ_0, function $\lambda\mapsto(\lambda_2^2-\lambda_0^2)^2$ has exactly one double root $\lambda=\lambda_0$. Since the right hand side in (5.16) is multiplied by the small parameter, by the Rouché theorem we obtain that equation (5.16) has exactly two roots (counting multiplicity) converging to λ_0 as $\varepsilon\to+0$.

Let us prove that roots $\lambda_\pm(\varepsilon,t)$ are jointly continuous w.r.t. ε and t. We choose ε_0, t_0 and fix $\delta>0$. Consider the contour

$$\Gamma_\delta:=\big\{\lambda\in\mathbb{C}:\,\operatorname{dist}(\lambda,\{\lambda_+(\varepsilon_0,t_0),\lambda_-(\varepsilon_0,t_0)\})=\delta\big\}.$$

Since this contour contains no zeroes of the function $(\lambda^2-\lambda_0^2)-\varepsilon_0 h(\varepsilon_0,t_0,\lambda)$ and this function is holomorphic w.r.t. λ, we have the estimate

$$\big|(\lambda^2-\lambda_0^2)-\varepsilon_0 h(\varepsilon_0,t_0,\lambda)\big|\geqslant\delta_1>0,$$

where δ_1 is a constant independent of λ. On the other hand, since function $h(\varepsilon,t,\lambda)$ is holomorphic w.r.t. ε, t, λ, there exists $\delta_2>0$ such that for $|\varepsilon-\varepsilon_0|<\delta_2$, $|t-t_0|<\delta_2$ we have

$$\big|\varepsilon h(\varepsilon,t,\lambda)-\varepsilon_0 h(\varepsilon_0,t_0,\lambda)\big|<\delta_1\quad\text{for all}\quad\lambda\in\Gamma_\delta.$$

Hence, rewriting equation (5.16) as

$$(\lambda^2-\lambda_0^2)^2-\varepsilon_0 h(\varepsilon_0,t_0,\lambda)+\varepsilon h(\varepsilon,t,\lambda)-\varepsilon_0 h(\varepsilon_0,t_0,\lambda)=0$$

and applying Rouché theorem, we see that $\lambda_\pm(\varepsilon,t)$ are located inside contour Γ_δ and therefore,

$$|\lambda_\pm(\varepsilon,t)-\lambda_\pm(\varepsilon_0,t_0)|<\delta\quad\text{as}\quad|\varepsilon-\varepsilon_0|<\delta_2,\quad|t-t_0|<\delta_2.$$

The proof is complete. ∎

Let us study the asymptotic behavior of $\lambda_\pm(\varepsilon,t)$ as $\varepsilon\to+0$. Definition (5.12) of operator $\mathcal{A}_7$ implies

$$\begin{aligned}\mathcal{A}_8^{(\natural\sharp)}(\varepsilon,t,\lambda)=&\big(\mathcal{A}_5(\varepsilon,t,\lambda)\psi_0^\natural,\psi_0^\sharp\big)_{L_2(-\pi,\pi)}\\&-\varepsilon\big(\mathcal{A}_5(\varepsilon,t,\lambda)\mathcal{A}_6(\lambda^2)\mathcal{A}_5(\varepsilon,t,\lambda)\psi_0^\natural,\psi_0^\sharp\big)_{L_2(-\pi,\pi)}+O(\varepsilon^2)\end{aligned}\tag{5.17}$$

uniformly in t and λ. Definition (5.8) of operator $\mathcal{A}_5$ and the oddness of γ yield

$$\begin{aligned}&\big(\mathcal{A}_5(\varepsilon,t,\lambda)\psi_0^\pm,\psi_0^\pm\big)_{L_2(-\pi,\pi)}=\varepsilon t^2,\\&\big(\mathcal{A}_5(\varepsilon,t,\lambda)\psi_0^\pm,\psi_0^\mp\big)_{L_2(-\pi,\pi)}=\mp2itn+i\lambda\alpha_0(n).\end{aligned}\tag{5.18}$$

Hence, $\operatorname{tr}\mathcal{A}_8=O(\varepsilon)$, $h=O(\varepsilon)$ and it follows from (5.16) that

$$\lambda_\pm(\varepsilon,t)=\lambda_0+O(\varepsilon).\tag{5.19}$$

Then the second term in (5.17) can be slightly simplified:

$$\begin{aligned}&\big(\mathcal{A}_5(\varepsilon,t,\lambda)\mathcal{A}_6(\lambda^2)\mathcal{A}_5(\varepsilon,t,\lambda)\psi_0^\natural,\psi_0^\sharp\big)_{L_2(-\pi,\pi)}\\&\quad=\big(\mathcal{A}_5(0,t,\lambda_0)\mathcal{A}_6(\lambda_0^2)\mathcal{A}_5(0,t,\lambda_0)\psi_0^\natural,\psi_0^\sharp\big)_{L_2(-\pi,\pi)}+O(\varepsilon).\end{aligned}\tag{5.20}$$

Functions $u_\pm:=\mathcal{A}_6(\lambda_0)f_\pm$ solve equations (2.17) and satisfy the identities $(u_\pm,\psi_0^+)_{L_2(-\pi,\pi)}=0$, $(u_\pm,\psi_0^-)_{L_2(-\pi,\pi)}=0$. We also observe that thanks to the parity properties of γ and

$\psi_0^{\pm}$, the right hand side in (2.17) is either odd or even and the same is true for the solutions: function u_+ is odd, while function u_- is even. Therefore,

$$\big(\mathcal{A}_5(0,t,\lambda_0)\mathcal{A}_6(\lambda_0^2)\mathcal{A}_5(0,t,\lambda_0)\psi_0^{\natural},\psi_0^{\sharp}\big)_{L_2(-\pi,\pi)} = i\lambda_0\left(\left(2it\frac{d}{dx}+i\lambda_0\gamma\right)u_{\natural},\psi_0^{\sharp}\right)_{L_2(-\pi,\pi)}$$
$$= -\lambda_0^2\alpha_1^{(\natural\sharp)}(n) - \left(u_{\natural}, 2t\frac{d\psi_0^{\sharp}}{dx}\right)_{L_2(-\pi,\pi)} = -\lambda_0^2\alpha_1^{(\natural\sharp)}(n),\quad \alpha_1^{(\natural\sharp)}(n) := (\gamma u_{\natural},\psi_0^{\sharp})_{L_2(-\pi,\pi)}.$$

In view of the parity of $u_{\natural}$, γ and $\psi_0^{\sharp}$, we see that $\alpha_1^{(+-)}=\alpha_1^{(-+)}=0$. These formulae and (5.20), (5.18), (5.17) lead us to the asymptotics for $\mathcal{A}_8^{(\natural\sharp)}$:

$$\begin{aligned}
&\mathcal{A}_8^{(\pm\pm)}(\varepsilon,t,\lambda) = \varepsilon(t^2+\lambda_0\alpha_1^{(\pm\pm)}(n)) + O(\varepsilon^2),\\
&\mathcal{A}_8^{(\pm\mp)}(\varepsilon,t,\lambda) = \mp 2itn + i\lambda\alpha_0(n) + O(\varepsilon^2),
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{tr}\mathcal{A}_8 &= \varepsilon\Big(2t^2+\lambda_0^2\alpha_1^{(++)}(n)+\lambda_0^2\alpha_1^{(--)}(n)\Big)+O(\varepsilon^2),\\
\det\mathcal{A}_8 &= \lambda^2\alpha_0^2(n) - 4t^2n^2 + O(\varepsilon^2)\\
&= (\lambda^2-\lambda_0^2)\alpha_0^2(n)+\lambda_0^2\alpha_0^2(n)-4t^2n^2+O(\varepsilon^2).
\end{aligned}\tag{5.21}$$

Employing the above formulae, we rewrite equation (5.16) as a pair of equivalent ones:

$$\begin{aligned}
\lambda-\lambda_0 =& \frac{\varepsilon}{2(\lambda+\lambda_0)}\big(\operatorname{tr}\mathcal{A}_8-\varepsilon\alpha_0^2(n)\big)\\
&\pm\frac{\varepsilon}{2(\lambda+\lambda_0)}\sqrt{(\operatorname{tr}\mathcal{A}_8-\varepsilon\alpha_0^2(n))^2-4\big(\det\mathcal{A}_8-(\lambda^2-\lambda_0^2)\alpha_0^2(n)\big)},
\end{aligned}\tag{5.22}$$

and therefore, by (5.21), (5.19),

$$\lambda-\lambda_0 = \pm\frac{i\varepsilon}{4\lambda_0}\sqrt{\lambda_0^2\alpha_0^2(n)-4t^2n^2+O(\varepsilon^2)}+O(\varepsilon^2).$$

Substituting this formula into (5.22), we finally obtain

$$\begin{aligned}
\lambda-\lambda_0 =& \frac{\varepsilon^2}{4\lambda_0}\Big(2t^2+\lambda_0^2\alpha_1(n)-\alpha_0^2(n)\Big)\\
&\pm\frac{i\varepsilon}{4\lambda_0}\sqrt{\lambda_0^2\alpha_0^2(n)-4t^2n^2+O(\varepsilon^2)}\left(1\mp\frac{i\varepsilon}{8\lambda_0^2}\sqrt{\lambda_0^2\alpha_0^2(n)-4t^2n^2+O(\varepsilon^2)}\right)+O(\varepsilon^3)\\
=&\pm\frac{i\varepsilon}{4\lambda_0}\sqrt{\lambda_0^2\alpha_0^2(n)-4t^2n^2+O(\varepsilon^2)}\\
&+\frac{\varepsilon^2}{4\lambda_0}\left(\left(2-\frac{n^2}{2\lambda_0^2}\right)t^2-\frac{7\alpha_0^2(n)}{8}+\lambda_0^2\alpha_1(n)\right)+O(\varepsilon^3).
\end{aligned}$$

The obtained asymptotics is apriori true for both roots $\lambda_{\pm}(\varepsilon,t)$. But it is not clear apriori how to choose sign '$\pm$' in these asymptotics: whether different roots correspond to different signs or to a same sign. However, as identities (2.4) say, these roots are complex conjugate: $\lambda_-(\varepsilon,t)=\overline{\lambda_+(\varepsilon,t)}$. This is why in the above asymptotics different roots correspond to different signs and it proves (2.15). Suppose that $n^2+\varkappa_*<0$; then λ_0 is pure imaginary and $\lambda_0^2<0$. In this case $\lambda_0^2\alpha_0^2(n)-4t^2n^2<0$ for all values of t and asymptotics (2.15) can be simplified and we arrive at (2.18). If $n^2+\varkappa_*>0$, then the first term in the square root in (2.15) can vanish for certain values of t and we have to keep this asymptotics as it is.

We proceed to the case $\frac{1}{2}-C\varepsilon\leqslant|\tau|\leqslant\frac{1}{2}$. It splits into two subcases: $\frac{1}{2}-C\varepsilon\leqslant\tau\leqslant\frac{1}{2}$ and $-\frac{1}{2}\leqslant\tau\leqslant-\frac{1}{2}+C\varepsilon$. Consider the latter subcase. In eigenvalue equation (5.3) we change function ψ_ε as $\psi_\varepsilon(x)=e^{ix}\tilde{\psi}_\varepsilon(x)$. Then for $\tilde{\psi}_\varepsilon$ we obtain the same equation but $\mathcal{H}_0(\tau)$ is to be replaced by $\mathcal{H}_0(\tau+1)$. The eigenvalues remain the same. Hence, the

above subcases can be joined into one case: $-C\varepsilon \leqslant \tau - \frac{1}{2} \leqslant C\varepsilon$. Here we proceed as in (5.8), (5.9), (5.10), (5.11), (5.12), (5.13), (5.14), (5.15), (5.16) with some changes. The first change is the definition of parameter t and of functions $\psi_0^\pm$:

$$\tau = \frac{1}{2} + \varepsilon t, \quad |t| \leqslant C, \quad \psi_0^+(x) := \frac{e^{-inx}}{\sqrt{2\pi}}, \quad \psi_0^-(x) := \frac{e^{i(n+1)x}}{\sqrt{2\pi}}.$$

Then we arrive at equation (5.16) for the sought eigenvalues and Lemma 5.1 holds true. Asymptotics (5.17) is still true, while identities (5.18) are to be modified:

$$\begin{aligned}
&\big(\mathcal{A}_5(\varepsilon,t,\lambda)\psi_0^+,\psi_0^+\big)_{L_2(-\pi,\pi)} = 2nt + \varepsilon t^2,\\
&\big(\mathcal{A}_5(\varepsilon,t,\lambda)\psi_0^-,\psi_0^-\big)_{L_2(-\pi,\pi)} = -2(n+1)t + \varepsilon t^2,\\
&\big(\mathcal{A}_5(\varepsilon,t,\lambda)\psi_0^\pm,\psi_0^\mp\big)_{L_2(-\pi,\pi)} = \pm\lambda\alpha_0(n+1),
\end{aligned} \tag{5.23}$$

and therefore,

$$\operatorname{tr}\mathcal{A}_8 = -2t + O(\varepsilon), \quad \det\mathcal{A}_8 = -4n(n+1)t^2 + O(\varepsilon).$$

Equation (5.16) is equivalent to the pair of equations

$$\lambda^2 - \lambda_0^2 = \frac{\varepsilon}{2}\operatorname{tr}\mathcal{A}_8(\varepsilon,t,\lambda) \pm \frac{\varepsilon}{2}\sqrt{\operatorname{tr}^2\mathcal{A}_8(\varepsilon,t,\lambda) - 4\det\mathcal{A}_8(\varepsilon,t,\lambda)}. \tag{5.24}$$

Thanks to (5.23), we have

$$\operatorname{tr}^2\mathcal{A}_8(\varepsilon,t,\lambda) - 4\det\mathcal{A}_8(\varepsilon,t,\lambda) = (4n+2)t^2 + \lambda^2\alpha_0^2(n+1) + O(\varepsilon)$$

and therefore, this expression is non-zero for all t, ε and λ. Hence, the square root in (5.24) is holomorphic in ε, t, λ. As in Lemma 5.1, one can prove that each of equations (5.24) has the unique root converging to λ_0 as $\varepsilon \to +0$. Moreover, by the inverse function theorem for holomorphic functions, roots $\lambda_\pm(\varepsilon,t)$ are holomorphic w.r.t. ε and t. Asymptotics (5.23) and equations (5.24) imply asymptotics (2.19).

6 Convergence of the eigenvalues near the essential spectrum

In this section we prove Statement 1 of Theorem 2.6. In other words, we consider the eigenvalue equation

$$(\mathcal{H}_0 + \varkappa_* + i\varepsilon\lambda\gamma - \lambda^2)\psi = 0 \tag{6.1}$$

and we want to study its eigenvalues converging to the points of the essential spectrum as $\varepsilon \to +0$. It is convenient to split our study into two parts. We begin with studying the pure periodic equation corresponding to the case when the localized potential is absent.

6.1 Floquet theory

In this subsection we consider the ordinary differential equation

$$-u'' + (\varkappa_* + i\varepsilon\lambda\gamma - \lambda^2)u = 0, \quad x \in \mathbb{R}, \tag{6.2}$$

where λ is assumed to range in a compact set in the complex plane. We are interested in fundamental solutions of this equation.

Since equation (6.2) is periodic, we apply the standard Floquet theory to the describe its fundamental solutions. Let $\Phi_\varepsilon = \Phi_\varepsilon(x,\lambda)$, $\Psi_\varepsilon = \Psi_\varepsilon(x,\lambda)$ be the solutions to (6.2) satisfying the initial conditions

$$\Phi_\varepsilon(0,\lambda) = 1, \quad \Phi'_\varepsilon(0,\lambda) = 0, \quad \Psi_\varepsilon(0,\lambda) = 0, \quad \Psi'_\varepsilon(0,\lambda) = 1. \tag{6.3}$$

We then introduce the matrix

$$\mathrm{A}^\varepsilon(\lambda) := \begin{pmatrix} \mathrm{A}^\varepsilon_{11}(\lambda) & \mathrm{A}^\varepsilon_{12}(\lambda) \\ \mathrm{A}^\varepsilon_{21}(\lambda) & \mathrm{A}^\varepsilon_{22}(\lambda) \end{pmatrix} = \begin{pmatrix} \Phi_\varepsilon(2\pi,\lambda) & \Psi_\varepsilon(2\pi,\lambda) \\ \Phi'_\varepsilon(2\pi,\lambda) & \Psi'_\varepsilon(2\pi,\lambda) \end{pmatrix}.$$

Since the Wronskian of functions Φ_ε, Ψ_ε is equal to one at $x=0$, we have

$$\det \mathrm{A}^\varepsilon(\lambda) = 1. \tag{6.4}$$

Employing this identity, we find the eigenvalues of matrix $\mathrm{A}^\varepsilon(\lambda)$:

$$\mu^\varepsilon_\pm(\lambda) = \frac{1}{2}\left(\mathrm{A}^\varepsilon_{11}(\lambda) + \mathrm{A}^\varepsilon_{22}(\lambda) \pm z\sqrt{(\mathrm{A}^\varepsilon_{11}(\lambda) + \mathrm{A}^\varepsilon_{22}(\lambda))^2 - 4}\right), \tag{6.5}$$

$$\mu^\varepsilon_+\mu^\varepsilon_- = 1, \quad \mu^\varepsilon_+ + \mu^\varepsilon_- = \mathrm{A}^\varepsilon_{11} + \mathrm{A}^\varepsilon_{22}, \quad \mu^\varepsilon_- - \mu^\varepsilon_+ = -z\sqrt{(\mathrm{A}^\varepsilon_{11} + \mathrm{A}^\varepsilon_{22})^2 - 4}, \tag{6.6}$$

where the branch of the square root is fixed by the condition $\sqrt{1} = 1$, while $z = +1$ or $z = -1$. The eigenvectors of matrix A^ε can be chosen as $\begin{pmatrix} -\mathrm{A}^\varepsilon_{12} \\ \mathrm{A}^\varepsilon_{11} - \mu^\varepsilon_\pm \end{pmatrix}$, and by the well-known theorem on fundamental solutions for periodic ordinary differential equations, the functions

$$\psi^\varepsilon_\pm(x,\lambda) := -\mathrm{A}^\varepsilon_{12}(\lambda)\Phi_\varepsilon(x,\lambda) + \big(\mathrm{A}^\varepsilon_{11}(\lambda) - \mu^\varepsilon_\pm(\lambda)\big)\Psi_\varepsilon(x,\lambda)$$

are solutions to equation (6.2) satisfying the identities

$$\psi^\varepsilon_\pm(x,\lambda) = e^{x\ln\mu^\varepsilon_\pm(\lambda)}\psi^\varepsilon_{\pm,per}(x,\lambda), \tag{6.7}$$

where $\psi^\varepsilon_{\pm,per}$ are 2π-periodic functions in x and

$$\psi^\varepsilon_\pm(2\pi m,\lambda) = \big(\mu^\varepsilon_\pm(\lambda)\big)^m\psi^\varepsilon_\pm(0,\lambda), \quad {\psi^\varepsilon_\pm}'(2\pi m,\lambda) = \big(\mu^\varepsilon_\pm(\lambda)\big)^m{\psi^\varepsilon_\pm}'(0,\lambda), \quad m\in\mathbb{Z}.$$

The Wronskian of functions ψ^ε_+ and ψ^ε_- is given by the formula

$$\begin{vmatrix} \psi^\varepsilon_+(x,\lambda) & \psi^\varepsilon_-(x,\lambda) \\ {\psi^\varepsilon_+}'(x,\lambda) & {\psi^\varepsilon_-}'(x,\lambda) \end{vmatrix} = \begin{vmatrix} -\mathrm{A}^\varepsilon_{12}(\lambda) & -\mathrm{A}^\varepsilon_{12}(\lambda) \\ \mathrm{A}^\varepsilon_{11}(\lambda) - \mu^\varepsilon_+(\lambda) & \mathrm{A}^\varepsilon_{11}(\lambda) - \mu^\varepsilon_-(\lambda) \end{vmatrix} = \mathrm{A}^\varepsilon_{12}(\lambda)\big(\mu^\varepsilon_-(\lambda) - \mu^\varepsilon_+(\lambda)\big).$$

We employ functions $\psi^\varepsilon_\pm$ and their above described properties to solve the problem

$$\begin{gathered} -u''_\varepsilon + (\varkappa_* + i\varepsilon\lambda\gamma - \lambda^2)u_\varepsilon = f, \quad x\in\mathbb{R}, \\ u_\varepsilon(x,\lambda) \sim C_\pm\psi^\varepsilon_\pm(x,\lambda), \quad x\to\pm\infty, \quad C_\pm = const, \end{gathered} \tag{6.8}$$

and two problems on half-axes

$$\begin{gathered} -u''_\varepsilon + (\varkappa_* + i\varepsilon\lambda\gamma - \lambda^2)u_\varepsilon = f, \quad \pm x > 0, \\ u_\varepsilon(0,\lambda) = 0, \quad u_\varepsilon(x,\lambda) \sim C_\pm\psi^\varepsilon_\pm(x,\lambda), \quad x\to\pm\infty, \quad C_\pm = const. \end{gathered} \tag{6.9}$$

Here function f is assumed to belong to $L_2(\mathbb{R}, e^{|\vartheta|x}\,dx)$, where

$$L_2(\mathbb{R}, e^{\vartheta|x|}\,dx) = \Big\{u\in L_{2,loc}(\mathbb{R}) : \int\limits_{\mathbb{R}} e^{\vartheta|x|}|u(x)|^2\,dx < \infty\Big\},$$

where ϑ is the same as in (2.1).

It is straightforward to check that the solution to problem (6.8) is given by

$$u_\varepsilon(x,\lambda) = \mathcal{A}_9(\varepsilon,\lambda)f := \int\limits_{-\infty}^{x} G^\varepsilon_0(x,y,\lambda)f(y)\,dy + \int\limits_{x}^{+\infty} G^\varepsilon_0(y,x,\lambda)f(y)\,dy, \tag{6.10}$$

$$
\begin{aligned}
G_0^\varepsilon(x,y,\lambda) =&\frac{\psi_+^\varepsilon(x,\lambda)\psi_-^\varepsilon(y,\lambda)}{\mathrm{A}_{12}^\varepsilon(\lambda)\big(\mu_-^\varepsilon(\lambda)-\mu_+^\varepsilon(\lambda)\big)}\\
=&\frac{1}{\mu_-^\varepsilon(\lambda)-\mu_+^\varepsilon(\lambda)}\Big(\mathrm{A}_{12}^\varepsilon(\lambda)\Phi_\varepsilon(x,\lambda)\Phi_\varepsilon(y,\lambda)\\
&-(\mathrm{A}_{11}^\varepsilon(\lambda)-\mu_+^\varepsilon)\Psi_\varepsilon(x,\lambda)\Phi_\varepsilon(y,\lambda)\\
&-(\mathrm{A}_{11}^\varepsilon(\lambda)-\mu_-^\varepsilon)\Phi_\varepsilon(x,\lambda)\Psi_\varepsilon(y,\lambda)-\mathrm{A}_{21}^\varepsilon(\lambda)\Psi_\varepsilon(x,\lambda)\Psi_\varepsilon(y,\lambda)\Big).
\end{aligned}
\tag{6.11}
$$

The above formula makes sense provided $\mu_+^\varepsilon(\lambda)\neq\mu_-^\varepsilon(\lambda)$.

In the same way we solve problems (6.9). The solution to the problem on the half-line $(0,+\infty)$ is

$$
u_\varepsilon(x,\lambda)=\mathcal{A}_9^+(\varepsilon,\lambda)f:=\int\limits_0^x G_+^\varepsilon(x,y,\lambda)f(y)\,dy+\int\limits_x^{+\infty} G_+^\varepsilon(y,x,\lambda)f(y)\,dy,
\tag{6.12}
$$

$$
\begin{aligned}
G_+^\varepsilon(x,y,\lambda) =&G_0^\varepsilon(x,y,\lambda)-\frac{\psi_+^\varepsilon(x,\lambda)\psi_+^\varepsilon(y,\lambda)}{\mathrm{A}_{12}^\varepsilon(\lambda)\big(\mu_-^\varepsilon(\lambda)-\mu_+^\varepsilon(\lambda)\big)}=-\frac{\psi_+^\varepsilon(x,\lambda)\Psi_\varepsilon(y,\lambda)}{\mathrm{A}_{12}^\varepsilon(\lambda)}\\
=&\Phi_\varepsilon(x,\lambda)\Psi_\varepsilon(y,\lambda)-\frac{\mathrm{A}_{11}^\varepsilon(\lambda)-\mu_+^\varepsilon(\lambda)}{\mathrm{A}_{12}^\varepsilon(\lambda)}\Psi_\varepsilon(x,\lambda)\Psi_\varepsilon(y,\lambda).
\end{aligned}
\tag{6.13}
$$

The solution to the problem on the half-line $(-\infty,0)$ is determined by the formula

$$
u_\varepsilon(x,\lambda)=\mathcal{A}_9^-(\varepsilon,\lambda)f:=\int\limits_{-\infty}^x G_-^\varepsilon(x,y,\lambda)f(y)\,dy+\int\limits_x^{0} G_-^\varepsilon(y,x,\lambda)f(y)\,dy,
\tag{6.14}
$$

$$
\begin{aligned}
G_-^\varepsilon(x,y,\lambda) =&G_0^\varepsilon(x,y,\lambda)-\frac{\psi_-^\varepsilon(x,\lambda)\psi_-^\varepsilon(y,\lambda)}{\mathrm{A}_{12}^\varepsilon(\lambda)\big(\mu_-^\varepsilon(\lambda)-\mu_-^\varepsilon(\lambda)\big)}\\
=&-\Phi_\varepsilon(y,\lambda)\Psi_\varepsilon(x,\lambda)+\frac{\mathrm{A}_{11}^\varepsilon(\lambda)-\mu_-^\varepsilon(\lambda)}{\mathrm{A}_{12}^\varepsilon(\lambda)}\Psi_\varepsilon(x,\lambda)\Psi_\varepsilon(y,\lambda).
\end{aligned}
\tag{6.15}
$$

Formulae (6.12), (6.13), (6.14), (6.15) make sense provided $\mathrm{A}_{12}^\varepsilon(\lambda)\neq 0$.

6.2 Expansion for matrix $\mathrm{A}^\varepsilon(\lambda)$

Our next step is to study the behavior of matrix $\mathrm{A}^\varepsilon(\lambda)$ for small ε. It is clear that solutions Φ_ε, Ψ_ε to Cauchy problems (6.2), (6.3) are holomorphic in ε and λ in the sense of the norm in $C^2[b_1,b_2]$ for each bounded segment $[b_1,b_2]\subset\mathbb{R}$. Their Taylor expansions are of the form:

$$
\Phi_\varepsilon(x,\lambda)=\sum_{j=0}^{\infty}(i\varepsilon\lambda)^j\Phi_j(x,\lambda),\quad \Psi_\varepsilon(x,\lambda)=\sum_{j=0}^{\infty}(i\varepsilon\lambda)^j\Psi_j(x,\lambda),
\tag{6.16}
$$

where Φ_i, Ψ_i are the solutions to the Cauchy problems

$$
\Phi_0''+(\lambda^2-\varkappa_*)\Phi_0=0,\quad x\in\mathbb{R},\quad \Phi_0(0,\lambda)=1,\quad \Phi_0'(0,\lambda)=0,
\tag{6.17}
$$
$$
\Psi_0''+(\lambda^2-\varkappa_*)\Psi_0=0,\quad x\in\mathbb{R},\quad \Psi_0(0,\lambda)=0,\quad \Psi_0'(0,\lambda)=1,
$$
$$
\Phi_i''+(\lambda^2-\varkappa_*)\Phi_i=\gamma\Phi_{i-1},\quad x\in\mathbb{R},\quad \Phi_i(0,\lambda)=\Phi_i'(0,\lambda)=0,
\tag{6.18}
$$
$$
\Psi_i''+(\lambda^2-\varkappa_*)\Psi_i=\gamma\Psi_{i-1},\quad x\in\mathbb{R},\quad \Psi_i(0,\lambda)=\Psi_i'(0,\lambda)=0.
\tag{6.19}
$$

Problems (6.17) have explicit solutions:

$$
\Phi_0(x,\lambda)=\cos\kappa(\lambda)x,\quad \Psi_0(x,\lambda)=\frac{\sin\kappa(\lambda)x}{\kappa(\lambda)}.
\tag{6.20}
$$

As $\kappa(\lambda)=0$, the above formula for Ψ_0 should be understood in the sense of the limit as $\kappa(\lambda)\to 0$: $\Psi_0(x,\lambda_0)=x$. The same is for all similar singularities which will appear in what follows.

Problems (6.18), (6.19) can be also solved explicitly:

$$
\begin{aligned}
\Phi_i(x,\lambda)&=\frac{1}{\kappa(\lambda)}\int\limits_0^x \gamma(t)\Phi_{i-1}(t,\lambda)\sin\kappa(\lambda)(x-t)\,dt,\\
\Psi_i(x,\lambda)&=\frac{1}{\kappa(\lambda)}\int\limits_0^x \gamma(t)\Psi_{i-1}(t,\lambda)\sin\kappa(\lambda)(x-t)\,dt.
\end{aligned}
\tag{6.21}
$$

Since function γ is 2π-periodic and odd, we have:

$$
\begin{aligned}
\Phi_1(2\pi,\lambda)=&\frac{1}{\kappa(\lambda)}\int\limits_0^{2\pi}\gamma(t)\cos\kappa(\lambda)t\sin\kappa(\lambda)(2\pi-t)\,dt\\
=&\frac{1}{2\kappa(\lambda)}\int\limits_0^{2\pi}\gamma(t)\big(\sin 2\pi\kappa(\lambda)+\sin\kappa(\lambda)(2\pi-2t)\big)\,dt=\rho_0(\lambda),\\
\Phi_1'(2\pi,\lambda)=&\int\limits_0^{2\pi}\gamma(t)\cos\kappa(\lambda)t\cos\kappa(\lambda)(2\pi-t)\,dt=0.
\end{aligned}
\tag{6.22}
$$

In the same way we get that

$$
\Psi_1(2\pi,\lambda)=0,\quad \Psi_1'(2\pi,\lambda)=-\rho_0(\lambda). \tag{6.23}
$$

Thus,

$$
\begin{aligned}
\mathrm{A}_{11}^{\varepsilon}(\lambda)&=\cos 2\pi\kappa(\lambda)+i\varepsilon\lambda\rho_0(\lambda)-\varepsilon^2\lambda^2\rho_{11}^{(2)}(\lambda)-i\varepsilon^3\lambda^3\rho_{11}^{(3)}(\lambda)+O(\varepsilon^4),\\
\mathrm{A}_{12}^{\varepsilon}(\lambda)&=\frac{\sin 2\pi\kappa(\lambda)}{\kappa(\lambda)}-\varepsilon^2\lambda^2\rho_{12}^{(2)}(\lambda)-i\varepsilon^3\lambda^3\rho_{12}^{(3)}(\lambda)+O(\varepsilon^4),\\
\mathrm{A}_{21}^{\varepsilon}(\lambda)&=-\kappa(\lambda)\sin 2\pi\kappa(\lambda)-\varepsilon^2\lambda^2\rho_{21}^{(2)}(\lambda)-i\varepsilon^3\lambda^3\rho_{21}^{(3)}(\lambda)+O(\varepsilon^4),\\
\mathrm{A}_{22}^{\varepsilon}(\lambda)&=\cos 2\pi\kappa(\lambda)-i\varepsilon\lambda\rho_0(\lambda)-\varepsilon^2\lambda^2\rho_{22}^{(2)}(\lambda)-i\varepsilon^3\lambda^3\rho_{22}^{(3)}(\lambda)+O(\varepsilon^4),
\end{aligned}
\tag{6.24}
$$

where

$$
\begin{aligned}
\rho_{11}^{(j)}(\lambda)&:=\Phi_j(2\pi,\lambda),\quad \rho_{12}^{(j)}(\lambda):=\Psi_j(2\pi,\lambda),\\
\rho_{21}^{(j)}(\lambda)&:=\Phi_j'(2\pi,\lambda),\quad \rho_{22}^{(j)}(\lambda):=\Psi_j'(2\pi,\lambda).
\end{aligned}
\tag{6.25}
$$

In particular, it implies:

$$
\mathrm{A}_{11}^{\varepsilon}(\lambda)+\mathrm{A}_{22}^{\varepsilon}(\lambda)=2\cos 2\pi\kappa(\lambda)-\varepsilon^2\lambda^2\big(\rho_{11}^{(2)}(\lambda)+\rho_{22}^{(2)}(\lambda)\big)-i\varepsilon^3\lambda^3\big(\rho_{11}^{(3)}(\lambda)+\rho_{22}^{(3)}(\lambda)\big)+O(\varepsilon^4).
\tag{6.26}
$$

6.3 Convergence of the eigenvalues

In this subsection we prove that all the isolated eigenvalues near the essential spectrum are localized in ε-neighbourhoods of the points $\pm\sqrt{n^2+\varkappa_*}$, $n\in\mathbb{Z}_+$ or converge to $\pm\sqrt{\varkappa_*}$ as $\varepsilon\to+0$.

We choose a fixed compact set in the complex plane and consider the eigenvalues of $\mathcal{H}_\varepsilon$ staying in this set for all sufficiently small ε. Due to (2.5), all such eigenvalues satisfy the estimate

$$
\operatorname{dist}\big(\kappa(\lambda),\sigma(\mathcal{H}_0)\big)\leqslant C\varepsilon
$$

for some constant C independent of ε and λ. Assume also that λ satisfy additional conditions:

$$\operatorname{dist}\big(\kappa(\lambda), n\big) \geqslant \delta(\varepsilon), \quad n \in \mathbb{Z}\setminus\{0\}, \quad \operatorname{dist}\big(\kappa(\lambda), 0\big) \geqslant C > 0, \tag{6.27}$$

where C is some positive constant independent of ε and λ, $\delta(\varepsilon)$ is an arbitrary positive function such that $\delta(\varepsilon)\to 0$, $\varepsilon/\delta(\varepsilon)\to 0$ as $\varepsilon\to +0$.

We rewrite eigenvalue equation (6.1) as

$$-\psi'' + \big(i\varepsilon\lambda\gamma - \kappa^2(\lambda)\big)\psi = -V\psi, \quad x\in\mathbb{R}. \tag{6.28}$$

If ψ is an eigenfunction of $\mathcal{H}_\varepsilon$, the right hand side in the above equation belongs to $L_2(\mathbb{R}, e^{\vartheta|x|}\,dx)$.

For each λ we choose z in (6.5) so that function $\psi_\pm^\varepsilon$ decay exponentially as $x\to\pm\infty$. Then it is clear that operator $\mathcal{A}_9(\varepsilon,\lambda)$ is a bounded one from $L_2(\mathbb{R}, e^{\vartheta|x|}\,dx)$ into $W_2^2(\mathbb{R}, e^{|\vartheta|x}\,dx)$, where

$$W_2^2(\mathbb{R}, e^{\vartheta|x|}\,dx) = \Big\{u\in W_{2,loc}^2(\mathbb{R}) : \int\limits_{\mathbb{R}} e^{\vartheta|x|}\big(|u(x)|^2 + |u'(x)|^2 + |u''(x)|^2\big)\,dx < \infty\Big\}.$$

Denoting $f := -V\psi$, we rewrite equation (6.28) as

$$\psi = \mathcal{A}_9(\varepsilon,\lambda) f. \tag{6.29}$$

We substitute this formula into (6.28) to obtain

$$f = -V\mathcal{A}_9(\varepsilon,\lambda) f. \tag{6.30}$$

Operator $V\mathcal{A}_9$ is a bounded one in $L_2(\mathbb{R}, e^{\vartheta|x|}\,dx)$. Moreover, relations (6.27), (6.6), (6.10), (6.11), (6.16), (6.20), (6.24), (6.25), (6.26) allow us to expand integral kernel G_0^ε of operator $\mathcal{A}_9$ (see (6.10)) as $\varepsilon\to+0$, $\delta(\varepsilon)\to 0$. It implies the following representation:

$$V\mathcal{A}_9(\varepsilon,\lambda) = -V\mathcal{A}_{10}(\lambda) - \mathcal{A}_{11}(\varepsilon,\lambda), \tag{6.31}$$

where $\mathcal{A}_{11}(\lambda)$ is a bounded operator in $L_2(\mathbb{R}, e^{\vartheta|x|}\,dx)$, and its norm is of order $O(\varepsilon/\delta)$ uniformly in considered values of λ. Operator $\mathcal{A}_{10}(\lambda)$ is given by the formula

$$\begin{aligned}
&\mathcal{A}_{10}(\lambda) f := \int\limits_{-\infty}^{x} G_0^0(x,y,\lambda) f(y)\,dy + \int\limits_{x}^{+\infty} G_0^0(y,x,\lambda) f(y)\,dy, \\
&G_0^0(x,y,\lambda) = -\frac{e^{iz\kappa(\lambda)(x-y)}}{2iz\kappa(\lambda)}.
\end{aligned} \tag{6.32}$$

It is straightforward to check that function $u=\mathcal{A}_{10}(\lambda) f$ is the solution to the problem

$$-u'' - \kappa^2(\lambda) u = f, \quad x\in\mathbb{R}, \quad u(x,\lambda)\sim C_\pm(\lambda) e^{iz\kappa(\lambda)|x|}, \quad x\to\pm\infty, \tag{6.33}$$

with some constants $C_\pm(\lambda)$. Hereinafter the symbol '$\sim$' stands for the asymptotic behavior at infinity up to an exponentially small term.

Operator $\mathcal{A}_{10}(\lambda)$ is a bounded one in $W_2^2(\mathbb{R}, e^{\vartheta|x|}\,dx)$ and it is bounded uniformly in λ. Moreover, employing the fact that potential V decays as $e^{-\vartheta|x|}$, it is easy to make sure that operator $V\mathcal{A}_{10}(\lambda)$ is compact in $L_2(\mathbb{R}, e^{\vartheta|x|})$.

Lemma 6.1. *Operator $\big(I + V\mathcal{A}_{10}(\lambda)\big)$ has a bounded inverse in $L_2(\mathbb{R}, e^{\vartheta|x|}\,dx)$ and the inverse is bounded uniformly in λ.*

Proof. Consider the equation

$$f + V\mathcal{A}_{10}(\lambda) f = g, \quad g\in L_2(\mathbb{R}, e^{\vartheta|x|}\,dx), \tag{6.34}$$

and denote $u := \mathcal{A}_{10}(\lambda)f$. Due to problem (6.33), equation (6.34) is equivalent to the problem

$$-u'' + Vu - \kappa^2(\lambda)u = g, \quad x \in \mathbb{R}, \quad u(x,\lambda) \sim C_\pm(\lambda)e^{iz\kappa(\lambda)|x|}, \quad x \to \pm\infty. \tag{6.35}$$

Once we solve this problem, the solution to (6.34) is given by the identity

$$f = g - Vu. \tag{6.36}$$

We introduce the Jost functions for problem (6.35). These are functions $Y_1(x,\kappa(\lambda))$, $Y_2(x,\kappa(\lambda))$, which are the solutions to the equation

$$-Y'' + VY - \kappa^2(\lambda)Y = 0, \quad x \in \mathbb{R},$$

with the following behavior at infinity:

$$Y_1(x,\kappa(\lambda)) \sim e^{iz\kappa(\lambda)x}, \quad x \to +\infty, \quad Y_2(x,\kappa(\lambda)) \sim e^{-iz\kappa(\lambda)x}, \quad x \to -\infty.$$

Here the sign in the exponent is to be chosen so that function Y_1 decays exponentially as $x \to +\infty$ and Y_2 decays exponentially as $x \to -\infty$.

It is known [1, Ch. 2, Sec. 2.6.5] that the Wronskian of these functions is equal to $-2iz\kappa(\lambda)a(\kappa(\lambda))$, where $a(\cdot)$ is the transmission coefficient and it is non-zero for the considered values of $\kappa(\lambda)$.

The solution to problem (6.35) is given by the formula

$$\begin{aligned} u(x,\kappa(\lambda)) = \frac{i}{2z\kappa(\lambda)a(z\kappa(\lambda))}\Bigg(& Y_1(x,z\kappa(\lambda)) \int\limits_{-\infty}^{x} Y_2(t,z\kappa(\lambda))g(t)\,dt \\ & + Y_2(x,z\kappa(\lambda)) \int\limits_{x}^{+\infty} Y_1(t,z\kappa(\lambda))g(t)\,dt\Bigg). \end{aligned}$$

Recovering then f by (6.36), we obtain the explicit representation for the inverse operator $\big(I + V\mathcal{A}_{10}(\lambda)\big)^{-1}$. Then by straightforward calculations one can check easily that this operator is bounded in $L_2(\mathbb{R}, e^{\vartheta|x|}\,dx)$ uniformly in λ. $\square$

Employing (6.31), we rewrite equation (6.30) as

$$f + V\mathcal{A}_{10}(\lambda)f + \mathcal{A}_{11}(\varepsilon,\lambda)f = 0. \tag{6.37}$$

Since operator $\mathcal{A}_{11}$ is small and by Lemma 6.1 operator $I + V\mathcal{A}_{10}$ has a bounded inverse, we conclude that the above equation has the trivial solution only. It proves the absence of the eigenvalues of $\mathcal{H}_\varepsilon$ outside ε-neighbourhoods of points $\pm\sqrt{n^2 + \varkappa_*}$ and outside a fixed neighbourhood of $\pm\sqrt{\varkappa_*}$.

7 Existence and asymptotics of the eigenvalues near the essential spectrum

This section is devoted to the proof of the rest of Theorem 2.6. Here we study the eigenvalues of $\mathcal{H}_\varepsilon$ in the vicinities of points $n\sqrt{1 + \frac{\varkappa_*}{n^2}}$, $n \in \mathbb{Z} \setminus \{0\}$. We choose one of such points and denote it by λ_0. We observe that λ_0 can be either real (if $n^2 + \varkappa_* \geqslant 0$) or pure imaginary (if $n^2 + \varkappa_* < 0$). Throughout the subsection we assume that

$$\lambda_0 \neq 0, \quad \rho_0(\lambda_0) = \frac{1}{2n}\int\limits_0^{2\pi} \gamma(t)\sin 2\pi(n-t)\,dt = -\frac{1}{2n}\int\limits_0^{2\pi} \gamma(t)\sin 2\pi nt\,dt \neq 0. \tag{7.1}$$

In accordance with the results of Subsection 6.3, possible eigenvalues in the vicinity of λ_0 satisfy the inequality $|\kappa(\lambda)-n|\leqslant C\varepsilon$ for some sufficiently large C and this is why they can be represented as

$$\kappa(\lambda)=n+\frac{i\varepsilon\lambda_0\rho_0(\lambda_0)\sin 2\zeta}{2\pi},\quad n\in\mathbb{Z}\setminus\{0\},\tag{7.2}$$

where $\zeta\in\mathbb{C}$ is a new parameter possibly depending on ε. We assume that $0\leqslant\operatorname{Re}\zeta<\pi$, $|\operatorname{Im}\zeta|\leqslant C$ for some sufficiently large C. We introduce such representation for $\kappa(\lambda)$ since it simplifies essentially many technical details below.

The approach we employ here involves operators $\mathcal{A}_9$ introduced by formulae (6.10), (6.11) and we need to know whether this operator is bounded uniformly in ε or not. In order to solve this issue, we substitute formulae (6.24) into (6.4) and equate the coefficients at the like powers of ε. It yields

$$\lambda_0^2\rho_0^2(\lambda_0)=\lambda_0^2\big(\rho_{11}^{(2)}(\lambda_0)+\rho_{22}^{(2)}(\lambda_0)\big).\tag{7.3}$$

This formula and (6.26) imply

$$\begin{aligned}\big(\mathrm{A}_{11}^\varepsilon(\lambda)+\mathrm{A}_{22}^\varepsilon(\lambda)\big)^2-4=&-4\big(\sin^2 2\pi\kappa(\lambda)+\lambda_0^2\rho_0^2(\lambda_0)\varepsilon^2\big)+O(\varepsilon^3)\\ =&-4\varepsilon^2\lambda_0^2\rho_0^2(\lambda_0)\cos^2 2\zeta+O(\varepsilon^3).\end{aligned}\tag{7.4}$$

This formula, identity (7.2), asymptotics (6.24), (6.26) and formula for $\mu_-^\varepsilon-\mu_+^\varepsilon$ in (6.6) with $z=1$ imply that

$$\mathrm{A}_{12}^\varepsilon(\lambda)=\frac{i\varepsilon}{n}\lambda_0\rho_0(\lambda_0)\sin 2\zeta+O(\varepsilon^2),\quad \mathrm{A}_{21}^\varepsilon(\lambda)=-i\varepsilon n\lambda_0\rho_0(\lambda_0)\sin 2\zeta+O(\varepsilon^2),\tag{7.5}$$

$$\begin{aligned}&\mu_\pm^\varepsilon(\lambda)=1\pm i\varepsilon\lambda_0\rho_0(\lambda_0)\cos 2\zeta+O(\varepsilon^2),\\ &\mu_-^\varepsilon(\lambda)-\mu_+^\varepsilon(\lambda)=-2i\varepsilon\lambda_0\rho_0(\lambda_0)\cos 2\zeta+O(\varepsilon^2),\\ &\mathrm{A}_{11}^\varepsilon(\lambda)-\mu_\pm^\varepsilon(\lambda)=i\varepsilon\lambda_0\rho_0(\lambda_0)(1\mp\cos 2\zeta)+O(\varepsilon^2).\end{aligned}\tag{7.6}$$

Formulae (7.6) are true provided

$$|\cos 2\zeta|\geqslant\delta>0\tag{7.7}$$

for some fixed $\delta>0$. To obtain (7.6), we also employed the identity

$$\sqrt{\lambda_0^2\rho_0^2(\lambda_0)\cos^2 2\zeta}=\lambda_0\rho_0(\lambda_0)\cos 2\zeta\tag{7.8}$$

while calculating the square roots in the definition of $\mu_\pm^\varepsilon$. This identity does not fit our choice of the branch of the square root (6.5), (6.6). But at the same time, such identity allows us to glue two branches of the square root since $\cos 2\zeta$ ranges in the whole complex plane, not just in the upper or lower half-plane. Such two branches are described by parameter z in (6.5), (6.6). This is why, while obtaining identities (7.6), we let $z=1$ since now the branches of the square root are glued by using parameter ζ.

In view of formulae (7.5), (7.6) operator $\mathcal{A}_9$ is bounded uniformly in ε provided condition (7.7) is satisfied. This is why in what follows we consider two main cases. In the first case we consider the eigenvalues such that the associated values of ζ (see (7.2)) satisfy condition (7.7) for some $\delta>0$ and sufficiently small ε; δ is independent of ε. In the second case we consider the eigenvalues such that the associated parameter ζ satisfies

$$\cos 2\zeta\to 0,\quad \varepsilon\to+0.\tag{7.9}$$

7.1 Case (7.7).

At the first step we need to know the expansion of $\mu_\pm^\varepsilon$ up to $O(\varepsilon^3)$. By (6.20), (6.26), (7.2), (7.3) we obtain:

$$A_{11}^\varepsilon(\lambda)+\mathrm{A}_{22}^\varepsilon(\lambda)=2-\varepsilon^2\lambda_0^2\rho_0^2(\lambda_0)\cos^2 2\zeta-i\varepsilon^3\rho_*(\zeta)+O(\varepsilon^4),\tag{7.10}$$

$$
\begin{aligned}
\rho_*(\zeta) :=& \lambda_0^3\big(\rho_{11}^{(3)}(\lambda_0) + \rho_{22}^{(3)}(\lambda_0)\big) \\
&+ \frac{\lambda_0 \rho_0(\lambda_0) \sin 2\zeta}{2\pi} \frac{d}{d\kappa}(\kappa^2 + \varkappa_*)(\rho_{11}^{(2)}(\lambda) + \rho_{22}^{(2)}(\lambda))\Big|_{\kappa=n}.
\end{aligned} \tag{7.11}
$$

In what follows we shall make use of the following auxiliary lemma.

Lemma 7.1. *Constant ρ_* is real for real ζ. The identities*

$$
\rho_*(\zeta) = \frac{\lambda_0 \rho_0(\lambda_0) \sin 2\zeta}{2\pi} \hat{\rho}, \quad \rho_{11}^{(2)}(\lambda_0) = \rho_{22}^{(2)}(\lambda_0) = \frac{\rho_0^2(\lambda_0)}{2}
$$

hold true.

We shall prove this lemma later. Now formulae (7.10) and (6.5) imply:

$$
\mu_\pm^\varepsilon(\lambda) = 1 \pm i\varepsilon\lambda_0\rho_0(\lambda_0)\cos 2\zeta - \frac{\varepsilon^2}{2}\left(\lambda_0^2\rho_0^2(\lambda_0) \pm \frac{\rho_*(\zeta)}{\lambda_0\rho_0(\lambda_0)\cos 2\zeta}\right) + O(\varepsilon^3). \tag{7.12}
$$

This identity, Lemma 7.1 and (7.2), (7.5), (7.6), (6.16) yield that operator $\mathcal{A}_9$ can be represented as

$$
\mathcal{A}_9(\varepsilon, \lambda) = \mathcal{A}_{12}(\zeta) + \varepsilon\mathcal{A}_{13}(\varepsilon, \zeta), \quad \mathcal{A}_{13}(\varepsilon, \zeta) = \mathcal{A}_{14}(\zeta) + \varepsilon\mathcal{A}_{15}(\varepsilon, \zeta), \tag{7.13}
$$

where

$$
\mathcal{A}_{12}(\zeta)f := \int\limits_{-\infty}^{x} G_1(x, y, \zeta)f(y)\,dy + \int\limits_{x}^{+\infty} G_1(y, x, \zeta)f(y)\,dy, \tag{7.14}
$$

$$
\mathcal{A}_{14}(\zeta)f := \int\limits_{-\infty}^{x} G_2(x, y, \zeta)f(y)\,dy + \int\limits_{x}^{+\infty} G_2(y, x, \zeta)f(y)\,dy, \tag{7.15}
$$

and the kernels of these operators are introduced as

$$
\begin{aligned}
G_1(x, y, \zeta) :=& -\frac{1}{2n\cos 2\zeta}\big(\sin 2\zeta \cos nx \cos ny - (1 - \cos 2\zeta)\sin nx \cos ny \\
&- (1 + \cos 2\zeta)\cos nx \sin ny + \sin 2\zeta \sin nx \sin ny\big) = \frac{\cos(nx + \zeta)\sin(ny - \zeta)}{n\cos 2\zeta}
\end{aligned}
$$

and

$$
G_2(x, y, \zeta) := -\frac{i\rho_*(\zeta)}{2\lambda_0^2\rho_0^2(\lambda_0)\cos^2 2\zeta}G_1(x, y, \zeta) + \frac{iG_3(x, y, \zeta)}{4\pi n^2\lambda_0\rho_0^2(\lambda_0)\cos^2 2\zeta}, \tag{7.16}
$$

$$
\begin{aligned}
G_3(x, y, \zeta) :=& -\lambda_0^2\rho_0(\lambda_0)\cos 2\zeta(2\pi n^2\rho_{12}^{(2)}(\lambda_0) - \rho_0^2(\lambda_0)\sin^2 2\zeta)\cos nx \cos ny \\
&+ \left(\pi n\lambda_0^{-1}\rho_*(\zeta) + n\rho_0^2(\lambda_0)\left(n\rho_0(\lambda_0) + \lambda_0^2\frac{d\rho_0}{d\kappa}(\lambda_0)\right)\cos 2\zeta \sin 2\zeta\right)\cos nx \sin ny \\
&- \left(\pi n\lambda_0^{-1}\rho_*(\zeta) - n\rho_0^2(\lambda_0)\left(n\rho_0(\lambda_0) + \lambda_0^2\frac{d\rho_0}{d\kappa}(\lambda_0)\right)\cos 2\zeta \sin 2\zeta\right)\sin nx \cos ny \\
&+ \lambda_0^2\rho_0(\lambda_0)\cos 2\zeta(2\pi\rho_{21}^{(2)}(\lambda_0) - \rho_0^2(\lambda_0)\sin^2 2\zeta)\sin nx \sin ny \\
&- 4\pi n\lambda_0^2\rho_0^2(\lambda_0)\cos 2\zeta\Big((\hat{\Phi}_1(y, \zeta) - n\hat{\Psi}_1(y, \zeta))\cos(nx + \zeta)\cos\zeta \\
&- (\hat{\Phi}_1(x, \zeta) - n\hat{\Psi}_1(x, \zeta))\sin(ny - \zeta)\sin\zeta\Big),
\end{aligned} \tag{7.17}
$$

$$
\hat{\Psi}_1(x, \zeta) := i\lambda_0\Psi_1(x, \lambda_0) + \frac{i\lambda_0\rho_0(\lambda_0)\sin 2\zeta}{2\pi n}\left(x\cos nx - \frac{\sin nx}{n}\right),
$$

$$
\hat{\Phi}_1(x, \zeta) := i\lambda_0\Phi_1(x, \lambda_0) - \frac{i\lambda_0\rho_0(\lambda_0)\sin 2\zeta}{2\pi}x\sin nx.
$$

Operators $\mathcal{A}_{13}(\varepsilon,\zeta)$, $\mathcal{A}_{15}(\varepsilon,\zeta)$ act from $L_2(\mathbb{R}, e^{\vartheta|x|}\,dx)$ into $W_2^2(\mathbb{R}, e^{\vartheta|x|}\,dx)$ and are bounded uniformly in ε and ζ. Moreover, operators $\mathcal{A}_{12}$, $\mathcal{A}_{13}$ are holomorphic in ζ.

Given a function $f \in L_2(\mathbb{R}, e^{\vartheta|x|}\,dx)$, the above representations for G_1 imply that function $u = \mathcal{A}_{12} f$ solves the equation

$$-u'' - n^2 u = f, \quad x \in \mathbb{R}, \tag{7.18}$$

and behaves at infinity as

$$\begin{aligned} &u(x,\zeta) \sim C_+(\zeta)\cos(nx+\zeta), \quad x \to +\infty,\\ &u(x,\zeta) \sim C_-(\zeta)\sin(nx-\zeta), \quad x \to -\infty, \end{aligned} \tag{7.19}$$

where $C_\pm(\zeta)$ are some constants.

We consider eigenvalue equation (6.28) and proceed as in Subsection 6.3, see equations (6.29), (6.30), (6.37). At that, we employ (7.13), (7.14) instead of (6.31), (6.32). It leads us to the equivalent equation:

$$f + V\mathcal{A}_{12}(\zeta) f + \varepsilon V \mathcal{A}_{13}(\varepsilon,\zeta) f = 0, \tag{7.20}$$

where f is introduced by (6.29).

Our next step is to study the inverse operator for $I + V\mathcal{A}_{12}(\zeta)$. In order to do it, we introduce the auxiliary function:

$$W(\zeta) := \operatorname{Re}\big(a(n)e^{2i\zeta} - b(n)\big) = |a(n)|\cos\big(2\zeta + \theta(n)\big) - b_{\mathrm{r}}(n), \tag{7.21}$$

where, we recall, $a(n)$, $\theta(n)$ and $b_{\mathrm{r}}(n)$ were defined in (2.22), (2.23), (2.24).

Lemma 7.2. *Operator $(I + V\mathcal{A}_{12}(\zeta))$ has a bounded inverse in $L_2(\mathbb{R}, e^{\vartheta|x|}\,dx)$ for all ζ except the zeroes of function $W_\pm(\zeta)$. The zeroes of function $W(\zeta)$ are $\zeta_\pm$.*

Assume that $|a_{\mathrm{i}}(n)| \neq |b_{\mathrm{r}}(n)|$. Then $\cos 2\zeta_\pm \neq 0$ and ζ is close to $\zeta_\pm$, the inverse operator $(I + V\mathcal{A}_{12}(\zeta))^{-1}$ has the simple pole:

$$\big(I + V\mathcal{A}_{12}(\zeta)\big)^{-1} = \frac{1}{\zeta - \zeta_\pm}\mathcal{A}_{16}^\pm + \mathcal{A}_{17}^\pm(\zeta), \tag{7.22}$$

$$\mathcal{A}_{16}^\pm g := V X_\pm \mathcal{A}_{18}^\pm g, \quad \mathcal{A}_{18}^\pm(\zeta) g := \Upsilon_\pm(n)\int\limits_{\mathbb{R}} X_\pm(t) g(t)\,dt, \tag{7.23}$$

and $\mathcal{A}_{17}^\pm(\zeta)$ is a bounded operator in $L_2(\mathbb{R}, e^{\vartheta|x|}\,dx)$ holomorphic in ζ close to $\zeta_\pm$.

Proof. Let us first prove that $\cos 2\zeta_\pm \neq 0$. Indeed, it follows from (2.25) that

$$\begin{gathered} \cos 2\zeta_\pm = \cos\left(\theta(n) \pm \arccos\frac{b_{\mathrm{r}}(n)}{|a(n)|}\right) = \frac{a_{\mathrm{r}}(n)b_{\mathrm{r}}(n) \mp a_{\mathrm{i}}(n)\sqrt{|a(n)|^2 - b_{\mathrm{r}}^2(n)}}{|a(n)|^2},\\ a_{\mathrm{r}}^2(n)b_{\mathrm{r}}^2(n) - a_{\mathrm{i}}^2(n)(|a(n)|^2 - b_{\mathrm{r}}^2(n)) = \big(b_{\mathrm{r}}^2(n) - a_{\mathrm{i}}^2(n)\big)|a(n)|^2 \neq 0. \end{gathered}$$

Hence, $\cos 2\zeta_\pm \neq 0$.

Now we proceed as in the proof of Lemma 6.1. Given $g \in L_2(\mathbb{R}, e^{\vartheta|x|}\,dx)$, we consider the equation

$$f + V\mathcal{A}_{12}(\zeta) f = g. \tag{7.24}$$

Denote $u := \mathcal{A}_{12}(\zeta) f$. It follows from (7.18), (7.19) that function u behaves at infinity in accordance with (7.19) and solves the equation

$$-u'' + Vu - n^2 u = g, \quad x \in \mathbb{R}. \tag{7.25}$$

The associated solution to equation (7.24) is given by the identity (6.36).

In order to solve problem (7.25), we introduce the associated Jost functions Y_1, Y_2, see (2.20), (2.21), (2.22), (2.23). Then the functions

$$X_1(x,\zeta) := 2\operatorname{Re} e^{i\zeta} Y_1(x,n), \quad X_2(x,\zeta) := -2i \operatorname{Im} e^{i\zeta} Y_2(x,n),$$

satisfy (7.19). Their Wronskian is equal to $4inW(\zeta)$. We use these functions as the fundamental solutions in order to solve equation (7.25):

$$u(x,\zeta) = -\frac{1}{4inW(\zeta)}\Bigg(\int\limits_{-\infty}^{x} X_1(x,\zeta)X_2(t,\zeta)g(t)\,dt + \int\limits_{x}^{+\infty} X_2(t,\zeta)X_1(x,\zeta)g(t)\,dt\Bigg), \tag{7.26}$$

The only singularities of u are at the zeroes of function $W(\zeta)$. These zeroes can be found explicitly and they are given by (2.25). Thanks to identity (2.23), we have $\frac{|b_{\mathrm{r}}(n)|}{|a(n)|} < 1$ and therefore, the zeroes of W satisfy $2\zeta_\pm + \theta(n) \neq 0$, $2\zeta_\pm + \theta(n) \neq \pi$. It yields that $W'(\zeta_\pm) = -|a(n)|\sin(2\zeta_\pm + \theta(n)) \neq 0$ and therefore, $2\zeta_\pm$ are simple zeroes. Hence, the function $W^{-1}(\zeta)$ has a simple pole at $\zeta_\pm$:

$$W^{-1}(\zeta) = \frac{1}{W'(\zeta_\pm)(\zeta-\zeta_\pm)} + \tilde{W}_\pm(\zeta), \tag{7.27}$$

where function $\tilde{W}_\pm(\zeta)$ is holomorphic in the vicinity of $\zeta_\pm$.

Since the Wronskian of X_1, X_2 vanishes at $\zeta = \zeta_\pm$, functions X_1, X_2 are linearly dependent. Comparing their behavior as $x \to +\infty$ and employing (2.22), we obtain that

$$X_2(x,\zeta_\pm) = \big(\overline{a(n)}e^{-2i\zeta_\pm} - b(n)\big)X_1(x,\zeta_\pm),$$

and thanks to the equation $W(\zeta_\pm) = 0$ and (7.21) we have

$$|a(n)|e^{-i(\zeta_\pm+\theta(n))} - b(n) = -i\big(|a(n)|\sin(2\zeta_\pm + \theta(n)) + b_{\mathrm{i}}(n)\big).$$

We substitute these identities and (7.27) into (7.26), (6.36) and arrive at (7.22), (7.23). ☐

Let us study equation (7.20). We are interested in ζ, for which this equation has a non-trivial solution. If ζ is separated from $\zeta_\pm$, by Lemma 7.2, the operator $(I+V\mathcal{A}_{12}(\zeta))$ has a bounded inverse and therefore, the same is true for $\big(I + V\mathcal{A}_{12}(\zeta) + \varepsilon V\mathcal{A}_{13}(\varepsilon,\zeta)\big)$ for sufficiently small ε. In this case, equation (7.20) has the trivial solution only. Thus, the sought values of ζ converge to $\zeta_\pm$ as $\varepsilon \to +0$. We consider one of points $\zeta_\pm$ and assume that it satisfies condition (7.7) and ζ ranges in a small vicinity of $\zeta_\pm$. Here we apply the modified Birman-Schwinger principle suggested in [16] in form proposed in [2], [4].

We apply the inverse operator $\big(I + V\mathcal{A}_{12}(\zeta)\big)^{-1}$ to equation (7.20) and we employ then representation (7.22):

$$f + \frac{\varepsilon}{\zeta-\zeta_\pm}\mathcal{A}_{16}^{\pm}V\mathcal{A}_{13}(\varepsilon,\zeta)f + \varepsilon\mathcal{A}_{17}^{\pm}V\mathcal{A}_{13}(\varepsilon,\zeta)f = 0. \tag{7.28}$$

Thanks to properties of operators $\mathcal{A}_{13}$, $\mathcal{A}_{16}^{\pm}$, $\mathcal{A}_{17}^{\pm}$, operator $\varepsilon\mathcal{A}_{17}^{\pm}V\mathcal{A}_{13}$ has a small norm and therefore, the inverse operator

$$\mathcal{A}_{19}^{\pm}(\varepsilon,\zeta) := \big(I + \varepsilon\mathcal{A}_{17}^{\pm}V\mathcal{A}_{13}(\varepsilon,\zeta)\big)^{-1} \tag{7.29}$$

is well-defined as a bounded operator in $L_2(\mathbb{R}, e^{\vartheta|x|}\,dx)$. It is holomorphic in ζ. We apply this operator to (7.28):

$$f + \frac{\varepsilon}{\zeta-\zeta_\pm}\mathcal{A}_{19}^{\pm}(\varepsilon,\zeta)\mathcal{A}_{16}^{\pm}V\mathcal{A}_{13}(\varepsilon,\zeta)f = 0 \tag{7.30}$$

and substitute definition (7.23) of operator $\mathcal{A}_{16}^{\pm}$:

$$f + \frac{\varepsilon}{\zeta-\zeta_\pm}\mathcal{A}_{18}^{\pm}(\zeta)(V\mathcal{A}_{13}f)\mathcal{A}_{19}^{\pm}(\varepsilon,\zeta)VX_\pm = 0. \tag{7.31}$$

The constant $\mathcal{A}_{18}^{\pm}(\zeta)(V\mathcal{A}_{13}f)$ is non-zero since otherwise the above equation yields $f=0$. We apply the functional $\mathcal{A}_{18}^{\pm}V\mathcal{A}_{13}(\varepsilon,\zeta)$ to this equation and we get:

$$\zeta-\zeta_{\pm}+\varepsilon h_{\pm}(\zeta,\varepsilon)=0, \quad h_{\pm}(\zeta,\varepsilon):=\mathcal{A}_{18}^{\pm}(\zeta)V\mathcal{A}_{13}(\varepsilon,\zeta)\mathcal{A}_{19}^{\pm}(\varepsilon,\zeta)VX_{\pm}. \tag{7.32}$$

Function $h_{\pm}$ is bounded uniformly in ε and ζ and it is holomorphic in ζ. Then by the Rouché theorem we obtain that equation (7.32) has precisely one simple zero $\zeta_{\pm}^{\varepsilon}$ converging to $\zeta_{\pm}$ as $\varepsilon\to+0$ and

$$\zeta_{\pm}^{\varepsilon}=\zeta_{\pm}+O(\varepsilon). \tag{7.33}$$

Expanding operators $\mathcal{A}_{19}^{\pm}$ into the Neumann series and employing (7.12), we get:

$$h_{\pm}(\zeta,\varepsilon)=\mathcal{A}_{18}^{\pm}(\zeta)V\mathcal{A}_{14}(\zeta)VX_{\pm}+O(\varepsilon)$$

and it allows us to specify the asymptotics for $\zeta_{\pm}^{\varepsilon}$:

$$\zeta_{\pm}^{\varepsilon}=\zeta_{\pm}-\varepsilon\hat{\zeta}_{\pm}+O(\varepsilon^2), \tag{7.34}$$

$$\hat{\zeta}_{\pm}:=\Upsilon_{\pm}(n)\int\limits_{\mathbb{R}}X_{\pm}(t)V(t)(\mathcal{A}_{14}(\zeta_{\pm})VX_{\pm})(t)\,dt. \tag{7.35}$$

Lemma 7.3. *Constant $\hat{\zeta}_{\pm}$ satisfies identity (2.26).*

We shall prove this lemma later.

For $\zeta=\zeta_{\pm}^{\varepsilon}$ equation (7.20) has exactly one non-trivial solution, which is given by the identity:

$$f_{\pm}^{\varepsilon}=\mathcal{A}_{19}^{\pm}(\varepsilon,\zeta_{\pm}^{\varepsilon})VX_{\pm}.$$

Since we are interested in the eigenvalues of operator pencil $\mathcal{H}_{\varepsilon}$, the associated eigenfunction should belong to $W_2^2(\mathbb{R})$. Function $f_{\pm}^{\varepsilon}$ generates a non-trivial solution to equation (6.28) with λ defined by (7.2), (7.33) and this non-trivial solution is given by the formula $\psi_{\varepsilon}^{\pm}=\mathcal{A}_9(\varepsilon,\lambda)f_{\varepsilon}^{\pm}$. This solution behaves at infinity in accordance with (6.8). Hence, functions $\psi_{\varepsilon}^{\pm}$ should decay as $|x|\to\infty$ and this condition determine ζ. Namely, in view of formula (6.7) and the first formula in (6.6), it is the case if and only if

$$\operatorname{Re}\ln\mu_{\varepsilon}^{+}=-\operatorname{Re}\ln\mu_{\varepsilon}^{-}=\ln|\mu_{\varepsilon}^{+}|>0. \tag{7.36}$$

Asymptotics (7.12) and (7.34) imply that

$$|\mu_{\varepsilon}^{\pm}|^2=1-2\varepsilon\rho_0(\lambda_0)\cos 2\zeta_{\pm}\operatorname{Im}\lambda_0+O(\varepsilon^2) \quad \text{if } \lambda_0 \text{ is pure imaginary}$$

$$|\mu_{\varepsilon}^{\pm}|^2=1-\frac{\varepsilon^2\rho_*(\zeta_{\pm})}{\lambda_0\rho_0(\lambda_0)\cos 2\zeta_{\pm}}+O(\varepsilon^3) \quad \text{if } \lambda_0 \text{ is real.}$$

Hence, by formula (2.25) for $\zeta_{\pm}$ and the formula for $\rho_*(\zeta)$ in Lemma 7.1 we conclude that condition (7.36) is satisfied provided inequalities (2.28) hold true and is not satisfied if we have the opposite strict inequalities (2.30).

Therefore, the eigenvalues of operator pencil $\mathcal{H}_{\varepsilon}$ converging to λ_0 are associated with roots $\zeta_{\pm}$ in the sense of asymptotics (2.27). If for a chosen root condition (2.28) is satisfied, then operator pencil $\mathcal{H}_{\varepsilon}$ has one isolated eigenvalue λ_{ε} converging to λ_0 as $\varepsilon\to+0$. This eigenvalue is simple and has asymptotics (2.27), (2.29). If for a chosen root condition (2.30) holds true, then operator pencil $\mathcal{H}_{\varepsilon}$ has no eigenvalues converging to λ_0 and having asymptotics (2.27). And operator pencil $\mathcal{H}_{\varepsilon}$ can have no other eigenvalues converging to λ_0 except the above described ones.

In conclusion of this subsection we prove Lemmata 7.1, 7.3.

Proof of Lemma 7.1. We begin with calculating the first term in the right hand side in (7.11). Employing (6.25), (6.19), (6.21), (6.22), (6.23) and integrating by parts, we obtain:

$$\rho_{11}^{(3)}(\lambda_0)=-\frac{1}{n}\int\limits_0^{2\pi}\gamma(t)\Phi_2(t,\lambda_0)\sin nt\,dt=-\int\limits_0^{2\pi}\Phi_2(t,\lambda_0)\big(\Psi_1''(t,\lambda_0)+n^2\Psi_1(t,\lambda_0)\big)\,dt$$

$$=\rho_{11}^{(2)}(\lambda_0)\rho_0(\lambda_0)-\int\limits_0^{2\pi}\Psi_1(t,\lambda_0)\gamma(t)\Phi_1(t,\lambda_0)\,dt$$

and in the same way we get

$$\rho_{22}^{(3)}(\lambda_0)=-\rho_{22}^{(2)}(\lambda_0)\rho_0(\lambda_0)+\int\limits_0^{2\pi}\Psi_1(t,\lambda_0)\gamma(t)\Phi_1(t,\lambda_0)\,dt.$$

Hence,

$$\rho_{11}^{(3)}(\lambda_0)+\rho_{22}^{(3)}(\lambda_0)=\rho_0(\lambda_0)\big(\rho_{11}^{(2)}(\lambda_0)-\rho_{22}^{(2)}(\lambda_0)\big). \tag{7.37}$$

It follows from (6.21), (6.25), (6.20) that

$$\begin{aligned}\rho_{11}^{(2)}(\lambda_0)-\rho_{22}^{(2)}(\lambda_0)=&-\frac{1}{n}\int\limits_0^{2\pi}dx\int\limits_0^{x}\gamma(x)\gamma(t)(\sin nx\cos nt+\cos nx\sin nt)\sin n(x-t)\,dt\\ =&-\frac{1}{n}\int\limits_0^{2\pi}dx\int\limits_0^{x}\gamma(x)\gamma(t)(\cos^2 nt-\cos^2 nx)\,dt.\end{aligned}$$

Making the change of variables $x\mapsto 2\pi-x$, $t\mapsto 2\pi-t$ and employing the oddness of γ, we arrive at the identities $\rho_{11}^{(2)}(\lambda_0)-\rho_{22}^{(2)}(\lambda_0)=0$. Hence,

$$\rho_{11}^{(3)}(\lambda_0)+\rho_{22}^{(3)}(\lambda_0)=0, \tag{7.38}$$

while by (7.3) we get the desired formulae for $\rho_{ii}^{(2)}(\lambda_0)$, $i=1,2$.

Let us find the second term in the right hand side in (7.11). It follows from (6.25), (6.18), (6.19), (6.20), (6.21) that

$$(\kappa^2+\varkappa_*)\big(\rho_{11}^{(2)}(\lambda)+\rho_{22}^{(2)}(\lambda)\big)=\Big(\kappa+\frac{\varkappa_*}{\kappa}\Big)\int\limits_0^{2\pi}dx\,\gamma(x)\int\limits_0^{x}\gamma(t)\sin\kappa(x-t)\sin\kappa(2\pi-x+t)\,dt,$$

and

$$\frac{d}{d\kappa}(\kappa^2+\varkappa_*)\big(\rho_{11}^{(2)}(\lambda)+\rho_{22}^{(2)}(\lambda)\big)\Big|_{\kappa=n}=\Big(\frac{\varkappa_*}{n^2}-1\Big)S_3-\Big(n+\frac{\varkappa_*}{n}\Big)S_4, \tag{7.39}$$

$$S_3:=\int\limits_0^{2\pi}dx\,\gamma(x)\int\limits_0^{t}\gamma(t)\sin^2 n(x-t)\,dt,\quad S_4:=\int\limits_0^{2\pi}dx\,\gamma(x)\int\limits_0^{x}(\pi+t-x)\sin 2n(t-x)\,dt.$$

Employing the oddness of γ, by straightforward calculations we check that

$$\begin{aligned}S_3=&\frac{1}{2}\int\limits_0^{2\pi}dx\,\gamma(x)\int\limits_0^{x}\gamma(t)\,dt-\frac{1}{2}\int\limits_0^{2\pi}dx\,\gamma(x)\int\limits_0^{x}\gamma(t)\cos 2n(x-t)\,dt\\ =&-\frac{1}{2}\int\limits_0^{2\pi}dx\,\gamma(x)\sin 2nx\int\limits_0^{x}\gamma(t)\sin 2nt\,dt=-\frac{1}{4}\left(\int\limits_0^{2\pi}\gamma(x)\sin 2nx\,dx\right)^2=-n^2\rho_0^2(\lambda_0).\end{aligned}$$

Substituting (7.37), (7.38), (7.39) and the above formula into (7.11), we arrive at the statement of the lemma. ☐

Proof of Lemma 7.3. Let us calculate the integral in definition (7.35) of $\hat{\zeta}_\pm$. As it follows from the definition of operators $\mathcal{A}_9$ and $\mathcal{A}_{14}$, given a function $f\in L_2(\mathbb{R},e^{\vartheta|x|}\,dx)$,

function $\mathcal{A}_{14}f$ is the coefficient at ε in the expansion of $\mathcal{A}_9 f$ in ε. Since function $v_\varepsilon = \mathcal{A}_9 f$ solves the equation

$$-v_\varepsilon'' + (i\varepsilon\lambda\gamma - \kappa^2(\lambda))v_\varepsilon = f, \quad x \in \mathbb{R},$$

thanks to (7.2), function $v_1 = \mathcal{A}_{14}f$ solves the equation

$$\begin{aligned}
&-v_1'' - n^2 v_1 = -i\lambda_0 \left(\gamma - \frac{n\rho_0(\lambda_0)\sin 2\zeta}{\pi}\right) v_0, \quad x \in \mathbb{R},\\
&v_0 := \mathcal{A}_{12}(\zeta)f, \quad -v_0'' - n^2 v_0 = f, \quad x \in \mathbb{R}.
\end{aligned}$$

It follows from the definition of function $X_\pm$ and operator $\mathcal{A}_{12}(\zeta)$ that

$$X_\pm = \mathcal{A}_{12}(\zeta)VX_\pm, \tag{7.40}$$

and therefore, the function $v := \mathcal{A}_{14}(\zeta_\pm)VX_\pm$ solves the equation

$$v'' + n^2 v = i\lambda_0 \left(\gamma - \frac{n\rho_0(\lambda_0)\sin 2\zeta_\pm}{\pi}\right) X_\pm. \tag{7.41}$$

In view of the behavior of $X_\pm$ at infinity and formula (7.40) we also have

$$\begin{gathered}
X_\pm(x) \sim 2\cos(nx+\zeta_\pm), \quad x \to +\infty, \quad X_\pm(x) \sim 2\varpi_\pm, \quad x \to -\infty,\\
\int\limits_{\mathbb{R}} \frac{\sin(nx-\zeta_\pm)}{n\cos 2\zeta_\pm} V(x)X_\pm(x)\,dx = 2, \quad \int\limits_{\mathbb{R}} \frac{\cos(nx+\zeta_\pm)}{n\cos 2\zeta_\pm} V(x)X_\pm(x)\,dx = 2\varpi_\pm.
\end{gathered} \tag{7.42}$$

Expanding $\sin(nx-\zeta_\pm)$ and $\cos(nx+\zeta_\pm)$ in two last identities and solving the obtained system of equations, we find:

$$\begin{aligned}
&\int\limits_{\mathbb{R}} V(x)X_\pm(x)\cos nx\,dx = 2n(\varpi_\pm \cos\zeta_\pm + \sin\zeta_\pm),\\
&\int\limits_{\mathbb{R}} V(x)X_\pm(x)\sin nx\,dx = 2n(\varpi_\pm \sin\zeta_\pm + \cos\zeta_\pm).
\end{aligned}$$

Definitions (6.21) of Φ_1 and Ψ_1 imply

$$\begin{aligned}
&\Phi_1(x) - n\Psi_1(x) = \frac{1}{n}\int\limits_0^{2\pi} \gamma(t)\sin n(x-t)\big(\cos nt - \sin nt\big)\,dt,\\
&(\Phi_1(x) - n\Psi_1(x))'\cos(nx+\zeta) - (\Phi_1(x) - n\Psi_1(x))(\cos(nx+\zeta))'\\
&= \int\limits_0^{x} \gamma(t)\cos(nt+\zeta)\big(\cos nt - \sin nt\big)\,dt,\\
&(\Phi_1(x) - n\Psi_1(x))'\sin(nx-\zeta) - (\Phi_1(x) - n\Psi_1(x))(\sin(nx-\zeta))'\\
&= \int\limits_0^{x} \gamma(t)\sin(nt-\zeta)\big(\cos nt - \sin nt\big)\,dt.
\end{aligned} \tag{7.43}$$

We employ equation (7.41) and the equation for $X_\pm$ and integrate be parts for arbitrary $N \in \mathbb{N}$:

$$\begin{aligned}
&\int\limits_{-2\pi N}^{2\pi N} X_\pm(x)V(x)(\mathcal{A}_{14}(\zeta_\pm)VX_\pm)(x)\,dx = \int\limits_{-2\pi N}^{2\pi N} (X_\pm'' + n^2 X_\pm)v\,dx\\
&= (X_\pm' v - X_\pm v')\Big|_{-2\pi N}^{2\pi N} + i\lambda_0 \int\limits_{-2\pi N}^{2\pi N} \left(\gamma - \frac{n\rho_0(\lambda_0)\sin 2\zeta_\pm}{\pi}\right) X_\pm^2\,dx.
\end{aligned} \tag{7.44}$$

Using (7.1), (7.42), (7.43), and definition (7.15), (7.16), (7.17) of operator $\mathcal{A}_{14}$, by straightforward calculations we check that

$$
\begin{aligned}
&(X'_\pm v - X_\pm v')\Big|_{-2\pi N}^{2\pi N} \sim 4\lambda_0^2 \sin\zeta_\pm \int\limits_0^{2\pi N} \gamma(t)\cos(nt+\zeta_\pm)(\cos nt - \sin nt)\,dt \\
&- 4\varpi_\pm^2 \lambda_0^2 \cos\zeta_\pm \int\limits_{-2\pi N}^{0} \gamma(t)\sin(nt-\zeta_\pm)(\cos nt - \sin nt)\,dt \\
&- 4Nn\lambda_0^2\rho_0(\lambda_0)\sin 2\zeta_\pm(\sin\zeta_\pm + \varpi_\pm^2\cos\zeta_\pm)(\cos\zeta_\pm + \sin\zeta_\pm) + S_1^\pm = S_1^\pm.
\end{aligned}
$$

Substituting this identity into (7.44) and passing to the limit as $N \to +\infty$, we arrive at the statement of the lemma. ☐

7.2 Case (7.9)

Here we study the eigenvalues such that parameter ζ associated with the eigenvalues of $\mathcal{H}_\varepsilon$ converges to $\frac{\pi}{4} + \pi m$ for $m = 0, 1$. Now expansion (7.13) is no longer valid since $\cos 2\zeta$ is not separated from zero. This is why we have to modify our approach. The modification is based on the technique proposed in [23] in the form how it was realized in [2, Sect. 5].

Here it is more convenient replace (7.2) by

$$
\kappa(\lambda) = n + \frac{i\varepsilon\lambda_0\rho_0(\lambda_0)}{2\pi\xi} \tag{7.45}
$$

assuming that $|\xi|$ is close to 1.

Given $g \in L_2(\mathbb{R})$, we introduce the function

$$
v(x,\varepsilon,\eta,\xi) = v_\pm(x,\varepsilon,\eta,\xi), \quad \pm x > 0, \quad v_\pm := \mathcal{A}_9^\pm(\varepsilon,\lambda)e^{-\frac{\vartheta}{2}|x|}g. \tag{7.46}
$$

Here $\eta = \eta(\varepsilon,\xi)$ is an additional auxiliary parameter introduced as

$$
\eta(\varepsilon,\xi) := \frac{i}{2\varepsilon\lambda_0\rho_0(\lambda_0)}\sqrt{(\mathrm{A}_{11}^\varepsilon(\lambda) + \mathrm{A}_{22}^\varepsilon(\lambda))^2 - 4}, \tag{7.47}
$$

where λ is defined by (7.2). Thanks to (7.4), (7.9) parameter η tends to zero as $\varepsilon \to +0$. In the above formulae for $v_\pm$ this parameter is involved only in $\mu_\pm$:

$$
\mu_\pm^\varepsilon(\lambda) = \frac{\mathrm{A}_{11}^\varepsilon(\lambda) + \mathrm{A}_{22}^\varepsilon(\lambda)}{2} \mp i\varepsilon\lambda_0\rho_0(\lambda_0)\eta(\varepsilon,\xi). \tag{7.48}
$$

We assume that η ranges in the vicinity of the zero in the complex plane. And as in (7.8), it allows us to let $z = 1$ in (6.5) that leads us to formula (7.48).

It follows from definition (6.12), (6.13), (6.14), (6.15) of operators $\mathcal{A}_9^\pm$ that

$$
\begin{aligned}
&v_\pm(0,\varepsilon,\eta,\xi) = 0, \\
&v'_+(0,\varepsilon,\eta,\xi) = \frac{\mathrm{A}_{22}^\varepsilon(\lambda) - \mathrm{A}_{11}^\varepsilon(\lambda) - 2i\lambda_0\rho_0(\lambda_0)\varepsilon\eta}{2\mathrm{A}_{12}^\varepsilon(\lambda)} \int\limits_0^{+\infty} \Psi_\varepsilon(y,\lambda)e^{-\frac{\vartheta}{2}|y|}g(y)\,dy, \\
&v'_-(0,\varepsilon,\eta,\xi) = \frac{\mathrm{A}_{11}^\varepsilon(\lambda) - \mathrm{A}_{22}^\varepsilon(\lambda) - 2i\lambda_0\rho_0(\lambda_0)\varepsilon\eta}{2\mathrm{A}_{12}^\varepsilon(\lambda)} \int\limits_{-\infty}^{0} \Psi_\varepsilon(y,\lambda)e^{-\frac{\vartheta}{2}|y|}g(y)\,dy.
\end{aligned} \tag{7.49}
$$

We then introduce one more function $w(x,\varepsilon,\eta,\xi)$ as the solution to the problem

$$
\begin{gathered}
w'' - w = 0, \quad x \in \mathbb{R}\setminus\{0\}, \qquad w(+0,\varepsilon,\eta,\xi) = w(-0,\varepsilon,\eta,\xi), \\
w'(+0,\varepsilon,\eta,\xi) - w'(-0,\varepsilon,\eta,\xi) = v'_+(0,\varepsilon,\eta,\xi) - v'_-(0,\varepsilon,\eta,\xi), \\
w(x,\varepsilon,\eta,\xi) = O(e^{-|x|}), \quad x \to \pm\infty.
\end{gathered} \tag{7.50}
$$

This solution can be found explicitly:

$$w(x,\varepsilon,\eta,\xi)=-\frac{1}{2}\big(v_+'(0,\varepsilon,\eta,\xi)-v_-'(0,\varepsilon,\eta,\xi)\big)e^{\mp x},\quad \pm x>0.$$

Let $\chi=\chi(x)$ be an infinitely differentiable cut-off function taking values in $[0,1]$, equalling one as $|x|<1$ and vanishing as $|x|>2$. We construct solutions to equation (6.28) as

$$\psi(x,\varepsilon)=\chi(x)w(x,\varepsilon,\eta,\xi)+v(x,\varepsilon,\eta,\xi). \tag{7.51}$$

In view of the boundary conditions at zero for v and w, the above function belongs to $W_{2,loc}^2(\mathbb{R})$. We substitute (7.51) into (6.28), multiply the result by $e^{\frac{\vartheta}{2}|x|}$ and arrive at the equation for g:

$$g+\mathcal{A}_{20}(\varepsilon,\eta,\xi)g=0, \tag{7.52}$$

$$\mathcal{A}_{20}(\varepsilon,\eta,\xi)g:=e^{\frac{\vartheta}{2}|x|}\Big(Vv+\chi(i\varepsilon\lambda\gamma+\varkappa_*-\lambda^2-1+V)w-2\chi'w'-\chi''w\Big). \tag{7.53}$$

In the same way as in [2, Lm. 5.2], [5, Sect. 3.A] one can prove the following statement.

Lemma 7.4. *Equation (6.28) is equivalent to equation (7.52). Namely, each solution to equation (7.52) generates a solution to equation (6.28) by formula (7.51). And vice versa, for each solution to equation (6.28) there exists a solution to equation (7.52) satisfying (7.51). Operator $\mathcal{A}_{20}$ is compact as an operator in $L_2(\mathbb{R})$.*

The correspondence between the solutions to equations (7.52) and (6.28) can be described explicitly. Indeed, given a solution to equation (7.52), the corresponding solution to equation (6.28) is generated by formula (7.51). And given a solution ψ to equation (6.28), corresponding functions v, w, g are introduced as

$$\begin{aligned}
&w(x,\varepsilon,\eta,\xi)=\psi(0)e^{\mp x},\quad \pm x>0,\quad v(x,\varepsilon,\eta,\xi)=\psi(x)-\chi(x)w(x,\varepsilon,\eta,\xi),\\
&g=\big(-v''+(\varkappa_*+i\varepsilon\lambda\gamma-\lambda^2)v\big)e^{\frac{\vartheta}{2}|x|}\\
&\ \ =\big(-V\psi-2\chi'w'-\chi''w+(\varkappa_*+i\varepsilon\lambda\gamma-\lambda^2-1)\chi w\big)\,e^{\frac{\vartheta}{2}|x|},\quad x\in\mathbb{R}\setminus\{0\}.
\end{aligned} \tag{7.54}$$

Thanks to assumption (7.9), ξ tends to ±1 as $\varepsilon\to+0$. Employing then (6.24), (6.25), (6.26), we can expand kernels $G_\pm^\varepsilon$ introduced in (6.13), (6.15). It allows us to state that operator $\mathcal{A}_{20}$ is jointly holomorphic w.r.t. ε, η, ξ and

$$\mathcal{A}_{20}(\varepsilon,\eta,\xi)=\mathcal{A}_{21}(\xi)+\eta\mathcal{A}_{22}(\xi)+\varepsilon\left(i\lambda_0\left(\gamma-\frac{n\rho_0(\lambda_0)}{2\pi\xi}\right)\mathcal{A}_{23}(\xi)+\mathcal{A}_{24}(\xi)\right)+\varepsilon\mathcal{A}_{25}(\xi,\eta,\varepsilon),$$

where operator $\mathcal{A}_{25}$ is a bounded one in $L_2(\mathbb{R})$ and is jointly holomorphic w.r.t. ε, η, ξ. For small ε and η it satisfies the estimate $\mathcal{A}_{25}(\varepsilon,\eta,\xi)=O(\varepsilon+|\eta|)$ in the sense of the operator norm. Operators $\mathcal{A}_{21}$, $\mathcal{A}_{22}$, $\mathcal{A}_{24}$ are introduced in the same way as $\mathcal{A}_{20}$ but integral kernels $G_\pm^\varepsilon$ in the definition of operators $\mathcal{A}_9^\pm$ are to be replaced by the following ones:

$$G_+^{(21)}(x,y,\xi)=-G_-^{(21)}(y,x,\xi)=\Phi_0(x,\lambda_0)\Psi_0(y,\lambda_0)-n\xi\Psi_0(x,\lambda_0)\Psi_0(y,\lambda_0),$$

for operator $\mathcal{A}_{21}$,

$$G_+^{(22)}(x,y,\xi)=G_-^{(22)}(y,x,\xi)=-n\xi\,\Psi_0(x,\lambda_0)\Psi_0(y,\lambda_0),$$

for operator $\mathcal{A}_{22}$,

$$\begin{aligned}
G_+^{(23)}(x,y,\xi)=&-G_-^{(23)}(x,y,\xi)=(\cos nx-\xi\sin nx)\hat{\Psi}_1(y,\xi)\\
&+\left(\hat{\Phi}_1(x)-n\xi\hat{\Psi}_1(x)-\frac{i\sin nx}{2\pi n\lambda_0\rho_0(\lambda_0)}\left((n^2+\lambda_0^2)\rho_0^2(\lambda_0)\right.\right.\\
&\left.\left.+n\lambda_0^2\rho_0(\lambda_0)\frac{d\rho_0}{d\kappa}(n)-2\pi n^2\lambda_0^2\xi^2\rho_{12}^{(2)}(\lambda_0)\right)\right)\frac{\sin ny}{n},
\end{aligned} \tag{7.55}$$

$$\hat{\Phi}_1(x,\xi):=i\lambda_0\Phi_1(x,\lambda_0)-\frac{i\lambda_0\rho_0(\lambda_0)}{2\pi\xi}x\sin nx,$$

$$\hat{\Psi}_1(x,\xi):=i\lambda_0\Psi_1(x,\lambda_0)+\frac{i\lambda_0\rho_0(\lambda_0)}{2\pi n\xi}\left(x\cos nx-\frac{\sin nx}{n}\right).$$

for operator $\mathcal{A}_{24}$. In the corresponding formulae (7.53) for $\mathcal{A}_i$, $i=21,22,23$, one should choose $\lambda=\lambda_0$, $\varepsilon=0$. Operator $\mathcal{A}_{23}(\xi)$ is introduced by the formula $\mathcal{A}_{23}(\xi)g=\chi w$, where function w in defined via (7.49), (7.50) with $\varepsilon=0$, $\eta=0$, $\lambda=\lambda_0$.

Then we can rewrite equation (7.52) as

$$
\begin{gathered}
g+(\mathcal{A}_{21}(\xi)+\eta\mathcal{A}_{22}(\xi))g+\varepsilon\mathcal{A}_{27}(\xi,\eta,\varepsilon)g=0, \qquad (7.56)\\
\mathcal{A}_{27}(\xi,\eta,\varepsilon):=i\lambda_0\left(\gamma-\frac{n\rho_0(\lambda_0)}{2\pi\xi}\right)\mathcal{A}_{23}(\xi)+\mathcal{A}_{24}(\xi)+\mathcal{A}_{25}(\xi,\eta,\varepsilon).
\end{gathered}
$$

Consider the equation

$$
g+\big(\mathcal{A}_{21}(\xi)+\eta\mathcal{A}_{22}(\xi)\big)g=e^{-\frac{a}{4}|x|}f,\quad f\in L_2(\mathbb{R}). \tag{7.57}
$$

It is easy to see that an analogue of Lemma 7.4 is true for this equation and it is equivalent to the problem

$$
\begin{gathered}
-u''+Vu-n^2u=e^{-\frac{a}{4}|x|}f,\quad x\in\mathbb{R},\\
u(x)\sim C_\pm\big(\cos nx-\xi(1\pm\eta)\sin nx\big),\quad x\to\pm\infty,
\end{gathered} \tag{7.58}
$$

for some constants $C_\pm$. Solutions to this problems are related to that of equation (7.57) by the formula similar to (7.51), (7.54). Problem (7.58) can be solved explicitly in terms of functions Y_1, Y_2, see (2.21), (2.22), (2.23):

$$
\begin{aligned}
u(x,\xi,\eta)=-\frac{1}{\hat{W}(\xi,\eta)}\Bigg(&Y_+(x,\xi,\eta)\int\limits_{-\infty}^{x}Y_-(y,\xi,\eta)e^{-\frac{a}{4}|y|}f(y)\,dy\\
&+Y_-(x,\xi,\eta)\int\limits_{x}^{+\infty}Y_+(y,\xi,\eta)e^{-\frac{|a|}{4}|y|}f(y)\,dy\Bigg),
\end{aligned} \tag{7.59}
$$

where

$$
\begin{aligned}
Y_+(x,\xi,\eta)&:=\frac{1}{\sqrt{2}}\left(Y_1(x,n)+\overline{Y_1(x,n)}+i\xi(1+\eta)\left(Y_1(x,n)-\overline{Y_1(x,n)}\right)\right),\\
Y_-(x,\xi,\eta)&:=\frac{1}{\sqrt{2}}\left(Y_2(x,n)+\overline{Y_2(x,n)}-i\xi(1-\eta)\left(Y_2(x,n)-\overline{Y_2(x,n)}\right)\right),
\end{aligned} \tag{7.60}
$$

and $\hat{W}$ is the Wronskian of functions Y_+, Y_-:

$$
\hat{W}(\xi,\eta):=2n\big((1-\eta^2)(a_\mathrm{i}+b_\mathrm{i})\xi^2+2(b_\mathrm{r}+a_\mathrm{r}\eta)\xi+a_\mathrm{i}-b_\mathrm{i}\big).
$$

We can recover solution to equation (7.57) by formulae (7.54), where $\psi(0)$ is to be replaced by $u(0,\eta,\xi)$ and one should take $\varepsilon=0$, $\lambda=\lambda_0$. We denote by $\mathcal{A}_{26}(\eta,\xi)$ the obtained inverse operator:

$$
\mathcal{A}_{26}(\xi,\eta):=\big(I+\mathcal{A}_{21}(\xi)+\eta\mathcal{A}_{22}(\xi)\big)^{-1}.
$$

We rewrite equation (7.56) as

$$
g+\varepsilon\mathcal{A}_{26}(\xi,\eta)\mathcal{A}_{27}(\xi,\eta,\varepsilon)g=0. \tag{7.61}
$$

As it follows from (7.59), operator $\mathcal{A}_{26}(\xi,\eta)$ is a bounded one in $L_2(\mathbb{R})$ provided $\hat{W}(\xi,\eta)\neq0$. Since we assume that $\eta\to0$, $|\xi|\to1$ as $\varepsilon\to+0$, the above condition is equivalent to $\hat{W}(\pm1,0)\neq0$, that is, $a_\mathrm{i}\neq\pm b_\mathrm{r}$. Under this condition, operator $\mathcal{A}_{26}$ is bounded uniformly in η and ξ. It implies immediately that for small ε equation (7.61) has the trivial solution only and therefore, operator pencil $\mathcal{H}_\varepsilon$ has no eigenvalues satisfying condition (7.9).

Assume $a_\mathrm{i} = \mp b_\mathrm{r}$. We also assume that $b_\mathrm{i} \neq 0$. Then function $\hat{W}(\xi,\eta)$ has a root converging to ± 1 as $\eta \to +0$. If $a_\mathrm{i} + b_\mathrm{i} \neq 0$, this root is

$$\xi(\eta) := \begin{cases} \dfrac{\pm a_\mathrm{i} - a_\mathrm{r}\eta + \sqrt{b_\mathrm{i}^2 \mp 2a_\mathrm{r}a_\mathrm{i}\eta + (1+a_\mathrm{i}^2)\eta^2}}{(a_\mathrm{i}+b_\mathrm{i})(1-\eta^2)} & \text{if} \quad a_\mathrm{i} + b_\mathrm{i} \neq 0, \\ \dfrac{a_\mathrm{i}}{\pm a_\mathrm{i} - a_\mathrm{r}\eta} & \text{if} \quad a_\mathrm{i} + b_\mathrm{i} = 0. \end{cases}$$

We see that in both cases this root is holomorphic w.r.t. small s. As ξ is close to $\xi(\eta)$, function $\hat{W}(\xi,\eta)$ can be represented as

$$\begin{gathered} \hat{W}(\xi,\eta) = \big(\xi - \xi(\eta)\big)\breve{W}(\xi,\eta), \\ \breve{W}(\xi,\eta) := 2n\Big((1-\eta^2)(a_\mathrm{i}+b_\mathrm{i})(\xi+\xi(\eta)) + 2(b_\mathrm{r} + a_\mathrm{r}\eta)\Big), \end{gathered} \tag{7.62}$$

and $\breve{W}$ is jointly holomorphic w.r.t. ξ close to ± 1 and η close to zero. In particular,

$$\breve{W}(1,0) = 4nb_\mathrm{i} \neq 0. \tag{7.63}$$

As $\xi = \xi(\eta)$, functions Y_+ and Y_- are linearly dependent since their Wronskian $\hat{W}(\xi(\eta),\eta)$ vanishes. Comparing the behavior of these functions as $x \to +\infty$ (see (7.60), (2.21), (2.22)), we get that

$$\begin{aligned} &Y_-(x,\xi(\eta),\eta) = \Upsilon(\eta) Y_+(x,\xi(\eta),\eta), \\ &\Upsilon(\eta) := \frac{\big(1 - i\xi(\eta)(1-\eta)\big)b(n) + \big(1 + i\xi(\eta)(1-\eta)\overline{a(n)}\big)}{1 + i\xi(\eta)(1+\eta)} \\ &\qquad = \frac{(1-\eta^2)(a_\mathrm{r} - b_\mathrm{r})\xi^2(\eta) + 2(b_\mathrm{i} - a_\mathrm{i}\eta)\xi(\eta) + a_\mathrm{r} + b_\mathrm{r}}{1 + \xi^2(\eta)(1+\eta)^2}. \end{aligned} \tag{7.64}$$

Let $g_0 \in L_2(\mathbb{R})$, $g_0 = g_0(x,\eta)$ be the solution to equation (7.56) with $f = 0$ corresponding to function $Y_+(x,\xi(\eta),\eta)$ via formulae (7.54) with $\varepsilon = 0$, $\lambda = \lambda_0$. It follows from the definition of operators $\mathcal{A}_{21}$, $\mathcal{A}_{22}$, $\mathcal{A}_{26}$ and (7.56), (7.60), (7.62), (7.63), (7.64) that for ξ close to ± 1 and small η operator $\mathcal{A}_{26}$ satisfies the representation:

$$\begin{aligned} &\mathcal{A}_{26}(\xi,\eta)g = \frac{\mathcal{A}_{28}(\eta)g}{\xi - \xi(\eta)} g_0 + \mathcal{A}_{29}(\xi,\eta)g, \\ &\mathcal{A}_{28}(\eta)g := -\frac{\Upsilon(\eta)}{4nb_\mathrm{i}} \int\limits_{\mathbb{R}} Y_+(x,\xi(\eta),\eta) e^{-\frac{\vartheta}{2}|x|} g(x)\,dx, \end{aligned}$$

where $\mathcal{A}_{29}(\xi,\eta)$ is a compact operator in $L_2(\mathbb{R})$ holomorphic w.r.t. ξ and η. We substitute this representation into equation (7.61) to obtain

$$g + \frac{\mathcal{A}_{28}(\eta)\mathcal{A}_{27}(\xi,\eta,\varepsilon)g}{\xi - \xi(\eta)} g_0 + \varepsilon\mathcal{A}_{29}(\xi,\eta)\mathcal{A}_{27}(\xi,\eta,\varepsilon)g = 0.$$

Proceeding as in (7.28), (7.29), (7.30), (7.31), we arrive at the equation for ξ:

$$\begin{gathered} \xi - \xi(\eta) = -\varepsilon\mathcal{A}_{28}(\eta)\mathcal{A}_{27}(\xi,\eta,\varepsilon)\mathcal{A}_{30}(\xi,\eta,\varepsilon)g_0, \\ \mathcal{A}_{30}(\xi,\eta,\varepsilon) := (I + \varepsilon\mathcal{A}_{29}(\xi,\eta)\mathcal{A}_{27}(\xi,\eta,\varepsilon))^{-1}. \end{gathered} \tag{7.65}$$

This equation determines values of ξ, for which equation (7.52) has a non-trivial solution. This solution is unique up to a multiplicative constant and can be defined as $g = \mathcal{A}_{30}(\xi,\eta,\varepsilon)g_0$, where ξ is a solution to equation (7.65).

We recall that parameter η depends on λ and hence on ξ, see (7.47). Let us study the solvability of this equation. We begin with an auxiliary lemma.

Lemma 7.5. *The identity*

$$\mathcal{A}_{28}(\eta)\mathcal{A}_{27}(\xi,\eta,\varepsilon)\mathcal{A}_{30}(\xi,\eta,\varepsilon)g_0 = i\hat{\xi}_\pm + O(|\xi \mp 1| + |\eta| + \varepsilon)$$

holds true as $\varepsilon \to +0$, $\eta \to 0$, $\xi \to \pm 1$.

We shall prove this lemma later in the end of the section.

It follows from (7.10), Lemma 7.1, the identity $\xi^{-1} = \sin^2 2\zeta$ and the holomorphic dependence of $\mathrm{A}_{11}^\varepsilon$, $\mathrm{A}_{22}^\varepsilon$ on ε and λ that

$$\left(\mathrm{A}_{11}^\varepsilon(\lambda) + \mathrm{A}_{22}^\varepsilon(\lambda)\right)^2 - 4 = -4\varepsilon^2\lambda_0^2\rho_0^2(\lambda_0)(1-\xi^{-2}) - 2i\varepsilon^3\frac{\lambda_0\rho_0(\lambda_0)\hat{\rho}}{\pi\xi} - 4\varepsilon^4\lambda_0^2\rho_0^2(\lambda_0)\varrho(\xi,\varepsilon),$$

where $\varrho(\xi,\varepsilon)$ is a function jointly holomorphic w.r.t. ξ and ε. Hence,

$$\eta = -\sqrt{1 - \frac{1}{\xi^2} + \frac{2i\varepsilon\hat{\rho}}{\pi\lambda_0\rho_0(\lambda_0)\xi} + \varepsilon^2\varrho(\xi,\varepsilon)}.$$

We denote $\tilde{\xi} := \xi \mp 1$ and rewrite the above identity as

$$\tilde{\xi} = \left(\eta^2 - \frac{2i\varepsilon\hat{\rho}}{\pi\lambda_0\rho_0(\lambda_0)(\tilde{\xi}\pm 1)} - \varepsilon^2\varrho(\tilde{\xi}\pm 1,\varepsilon)\right)\frac{(\tilde{\xi}\pm 1)^2}{\tilde{\xi}\pm 2}.$$

By the inverse function theorem, this equation can be solved w.r.t. $\tilde{\xi} = \tilde{\xi}(\eta^2,\varepsilon)$, $\tilde{\xi}(0,0) = 0$ and the solution is unique and holomorphic in ε and η^2. We also have

$$\tilde{\xi}(\eta^2,\varepsilon) = -\frac{i\varepsilon\hat{\rho}}{\pi\lambda_0\rho_0(\lambda_0)} + O(\varepsilon^2 + |\eta|^2). \tag{7.66}$$

We substitute this formula into (7.65) and expand $\xi(\eta)$ into the Taylor series. It allows us to rewrite equation (7.65) as

$$-\frac{a_\mathrm{r}}{b_\mathrm{i}}\eta = i\varepsilon\left(\hat{\xi}_\pm - \frac{\hat{\rho}}{\pi\lambda_0\rho_0(\lambda_0)}\right) + \eta^2\varrho_1(\eta,\varepsilon) + \varepsilon\eta\varrho_2(\eta,\varepsilon) + \varepsilon^2\varrho_3(\eta,\varepsilon),$$

where ϱ_i, $i = 1,2,3$, are functions jointly holomorphic in η and ε. Constant $a_\mathrm{r}/b_\mathrm{i}$ is non-zero thanks to the identity $a_\mathrm{i} = \pm b_\mathrm{r}$ and (2.23). The obtained equation for η is solvable by the inverse function theorem and it has the unique root $\eta(\varepsilon)$ holomorphic in ε such that $\eta(0) = 0$. The associated solution to equation (7.65) is recovered by the formula $\xi = \pm 1 + \tilde{\xi}(\eta^2(\varepsilon),\varepsilon)$. We also have

$$\eta(\varepsilon) = -\frac{i\varepsilon b_\mathrm{i}}{a_\mathrm{r}}\left(\hat{\xi}_\pm - \frac{\hat{\rho}}{\pi\lambda_0\rho_0(\lambda_0)}\right) + O(\varepsilon^2) \tag{7.67}$$

and by (7.66) it implies for the associated value of ξ:

$$\frac{1}{\xi} = \frac{1}{\tilde{\xi}\pm 1} = \pm 1 + \frac{i\varepsilon\hat{\rho}}{\pi\lambda_0\rho_0(\lambda_0)} + O(\varepsilon^2). \tag{7.68}$$

Formulae (7.67), (7.48), (7.10) lead us to the asymptotics for $\mu_\varepsilon^+(\lambda)$:

$$\mu_\varepsilon^+(\lambda) = 1 + \varepsilon^2\frac{\lambda_0\rho_0(\lambda_0)}{c_0}\left(\hat{\xi}_\pm - \frac{\hat{\rho}}{\pi\lambda_0\rho_0(\lambda_0)}\right) + O(\varepsilon^3)$$

and hence,

$$|\mu_\varepsilon^+(\lambda)|^2 = 1 - 2\varepsilon^2\,\mathrm{Re}\,\frac{\lambda_0\rho_0(\lambda_0)b_\mathrm{i}}{a_\mathrm{r}}\left(\hat{\xi}_\pm - \frac{\hat{\rho}}{\pi\lambda_0\rho_0(\lambda_0)}\right) + O(\varepsilon^3).$$

Thus, as in (7.36), (2.28), (2.30), (2.30), operator pencil $\mathcal{H}_\varepsilon$ has the unique eigenvalue converging to λ_0 as $\varepsilon \to +0$ provided condition (2.31) is satisfied. This eigenvalue is simple and its asymptotics (2.27), (2.32) is due to (7.68), (7.45), (6.20). If condition (2.33) is satisfied, then operator pencil $\mathcal{H}_\varepsilon$ has no eigenvalues converging to λ_0.

Proof of Lemma 7.5. We first observe that the desired formula follows directly from the definition of operators $\mathcal{A}_{28}$, $\mathcal{A}_{27}$, $\mathcal{A}_{30}$ with

$$\begin{aligned}\hat{\xi}_\pm :=& - i\mathcal{A}_{28}(0)\mathcal{A}_{27}(\pm 1, 0, 0)\mathcal{A}_{30}(\pm 1, 0, 0)g_0(\cdot, 0)\\ =&\mathcal{A}_{28}(0)\left(\lambda_0\left(\gamma - \frac{n\rho_0(\lambda_0)}{2\pi\xi}\right)\mathcal{A}_{23}(\pm 1) - i\mathcal{A}_{24}(\pm 1)\right)g_0(\cdot, 0).\end{aligned}$$

It follows directly from the definition of g_0 that $g_0(\cdot, 0)$ corresponds to $X_\pm(x) := Y_+(x, \pm 1, 0)$ in the sense of formula (7.51). Function $X_\pm$ behaves at infinity as

$$\begin{aligned}&X_\pm(x) \sim \sqrt{2}(\cos nx \mp \sin nx), \quad x \to +\infty,\\ &X_\pm(x) \sim \sqrt{2}(a_{\mathrm{r}} \pm b_{\mathrm{i}})(\cos nx \mp \sin nx), \quad x \to -\infty,\\ &\int\limits_0^{+\infty} e^{-\frac{\vartheta}{2}|x|}\frac{\sin nx}{n}g_0(x, 0)\,dx = \sqrt{2}, \quad \int\limits_{-\infty}^{0} e^{-\frac{\vartheta}{2}|x|}\frac{\sin nx}{n}g_0(x, 0)\,dx = -\sqrt{2}(a_{\mathrm{r}} \mp b_{\mathrm{i}}).\end{aligned} \tag{7.69}$$

We also have

$$\Upsilon(0) = a_{\mathrm{r}} \mp b_{\mathrm{i}}. \tag{7.70}$$

Let v_0, w_0 be functions v and w associated with $X_\pm$ and g_0 in the sense of formulae (7.51), (7.54). By $v_{23} = v_{23}(x)$ we denote the function obtained by the formulae similar to (7.46), but kernels $G_\pm^\varepsilon$ in the integral operator are to be replaced by $G_\pm^{(23)}$. Let W_{23} be the solution to problem (7.50) with v replaced by v_{23}.

For arbitrary $N \in \mathbb{N}$ formula (7.69) implies

$$\begin{aligned}\hat{\xi}_\pm = -\frac{\Upsilon(0)}{4nb_{\mathrm{i}}}\lim_{N\to+\infty}\Bigg(&\lambda_0\int\limits_{-2\pi N}^{2\pi N} X_\pm\left(\gamma - \frac{n\rho_0(\lambda_0)}{\pi}\right)w_0\chi\,dx\\ &- i\int\limits_{-2\pi N}^{2\pi N} X_\pm\left(Vv_{23} + \left(-\frac{d^2}{dx^2} - n^2 + V\right)\chi w_{23}\right)\,dx\Bigg).\end{aligned}$$

Employing the equation for $X_\pm$ and integrating by parts in the second integral, we get:

$$\begin{aligned}&\int\limits_{-2\pi N}^{2\pi N} X_\pm\left(Vv_{23} + \left(-\frac{d^2}{dx^2} - n^2 + V\right)\chi w_{23}\right)\,dx\\ &\qquad = \int\limits_{-2\pi N}^{2\pi N} v_{23}\left(\frac{d^2}{dx^2} + n^2\right)X_\pm\,dx + X_\pm(0)\big(w_{23}'(+0) - w_{23}'(-0)\big)\\ &\qquad = \int\limits_{-2\pi N}^{2\pi N} X_\pm\left(\frac{d^2}{dx^2} + n^2\right)v_{23}\,dx - (X_\pm v_{23}' - X_\pm' v_{23})\big|_{-2\pi N}^{2\pi N}.\end{aligned}$$

Since function v_{23} is in fact the coefficient at ε in the expansion of v_ε w.r.t. ε and η, and v_ε solves the equation

$$-v_\varepsilon'' + (i\varepsilon\lambda\gamma - \kappa^2(\lambda))v_\varepsilon = e^{-\frac{\vartheta}{2}|x|}g_0,$$

function v_{23} satisfies the equation

$$v_{23}'' + n^2 v_{23} = i\lambda_0\left(\gamma - \frac{\rho_0(\lambda_0)}{\pi}\right)v_0.$$

Employing the above obtained formulae and $X_\pm = v_0 + \chi w_0$, we get:

$$\hat{\xi}_\pm = -\frac{\Upsilon(0)}{4nb_{\mathrm{i}}} \lim_{N\to+\infty} \left(\lambda_0 \int\limits_{-2\pi N}^{2\pi N} \left(\gamma - \frac{n\rho_0(\lambda_0)}{\pi} \right) X_\pm^2 \, dx + i(X_\pm v_{23}' - X_\pm' v_{23})\Big|_{-2\pi N}^{2\pi N} \right). \tag{7.71}$$

It follows from formulae (7.55), (7.69) and the definition of v_{23} that

$$\begin{aligned} v_{23}(x) \sim {} & (\cos nx \mp \sin nx) \int\limits_0^{+\infty} \hat{\Psi}_1(y,1) g_0(0,y)\, dy \\ & + 2\big(\hat{\Phi}_1(x) - n\hat{\Psi}_1(x) + iS_2 \sin nx\big), \quad x \to +\infty, \\ v_{23}(x) \sim {} & (\cos nx \mp \sin nx) \int\limits_{-\infty}^{0} \hat{\Psi}_1(y,1) g_0(0,y)\, dy \\ & - 2(a_{\mathrm{r}} \pm b_{\mathrm{i}})\big(\hat{\Phi}_1(x) - n\hat{\Psi}_1(x) + iS_2 \sin nx\big), \quad x \to +\infty, \end{aligned} \tag{7.72}$$

where C was introduced in the statement of the lemma.

Using (6.21), it is straightforward to check that

$$\begin{aligned} & (\cos nx \pm \sin nx)(\hat{\Phi}_1(x) - n\hat{\Psi}_1(x) + iS_2 \sin nx) \\ & - (\cos nx \mp \sin nx)'(\hat{\Phi}_1(x) - n\hat{\Psi}_1(x) + iS_2 \sin nx) \\ & = i\lambda_0 \int\limits_0^x \gamma(t)(1 - \sin 2nt)\, dt + inS_2 + \frac{i\lambda_0\rho_0(\lambda_0)}{\pi}(\sin^2 nx - nx). \end{aligned}$$

We substitute this formula and (7.72), (7.70) into (7.71) and arrive at the desired identity for $\hat{\xi}$. ☐

Acknowledgments

D.I.B. was partially supported by RFBR grant no. 15-31-20037-mol_ved_a and by the project "Asymptotic analysis of boundary value problems", FRA 2014, University of Sannio. G.C. is a member of GNAMPA (INDAM).

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